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ABSTRACT

This programed course was developed for use with elementary school teachers who are trying to improve their competence in mathematics. The thirty chapters include: Pre-number Ideas; Whole Numbers; An Introduction to Geometry: Points, Lines, and Planes; Factors and Primes; Introducing Rational Numbers; Measure of Area; and The Real Numbers. (MK)

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PROGRAMED BRIEF COURSE IN MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS

(Preliminary Edition)

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PROGRAMED BRIEF COURSE IN MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS

(Preliminary Edition)

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PREFACE

Preliminary Statement

This program is developed for use with elementary school teachers who are trying to imporve their competence in mathematics. Since the project is supported by the School Mathematics Study Group, the textbook SMSG Volume IX, *A Brief Course in Mathematics for Elementary School Teachers*, was used as the point of departure with the expectation that the programed materials could serve as added help for teachers who are using the text and/or the films developed to supplement it. Such a program, if indeed it is an effective learning device, may be used with the above mentioned textbook, with other similar textbooks or without any supplementary materials. However, research supports the thesis that learning is more likely when two or more media are used than when learners depend on one medium.

For the reasons stated above this program follows the SMSG Volume IX rather closely. It has been separated into units associated with the chapters in the text. The symbolism has been made to conform to that used in the book, although some ideas and concepts have been extended where necessary for elementary school teachers.

The Point of View Toward Programing

Although each member of the Writing Team was furnished with a copy of the first two volumes of *A Program for First Year Algebra* written by an SMSG Writing Team at Stanford University during the summer of 1963, it was not considered mandatory that the Writing Team for Elementary School Mathematics follow either the format or the philosophy of this particular approach to programed instruction.

The Team had little knowledge of the theory involved in programed instruction and no experience in writing programed material. However, many books regarding programing were made available to it before any programing was attempted.

These were read and discussed at group sessions and the project consultant, Dr. Robert W. Scofield, Head, Department of Psychology, Oklahoma State University, was of valuable assistance in setting up a procedure for writing the programs. The procedure finally agreed upon about January 1, 1965, may be described as follows:

1. Make an itemized outline of the subject matter to be covered for continuity and complete coverage.
2. Since the student is supposed to learn from a careful consideration of the frames of the program, each portion of the program is presented with little or no introductory remarks. When an introduction is included it will consist of a few sentences to relate the material to what has already been done and to give the student some idea of what to expect. It is true that the student must be told some things, but it is usually possible to include this in the frames with proper cueing. Definitions are usually restated for emphasis. We have not used the method of telling the students and then asking them to exhibit this knowledge in a series of frames, following these statements.

The program consists of two sorts of frames. The Skinnerian type and the Crowder type. This choice was made primarily because the Writing Team felt that this format would add to the appeal and the usefulness of the program as a whole. It was felt that some types of subject matter lend themselves to one approach and some to another. Decisions were made on these while the programs were being developed. The variety introduced into the programs was also considered useful.

3. The program for each chapter is divided into several sub-programs to conform to natural divisions of the subject matter. For example, one Sub-Program in Chapter 4 deals with Numeration Systems to Other Bases and another Sub-Program deals with Addition and Multiplication with Numerals to the Base Seven and to the Base Five.

   In addition to this some of the sub-programs are followed by short statements summarizing the content covered. This matter of pulling the material together at frequent intervals is considered a good programing technique. Another reason for this is to exhibit the structure of mathematics to the student. Otherwise the realization of the existing structure of the subject would be left to the student and it is doubtful whether this would, in fact, take place while the student is immersed in a long list of frames with no distinctive divisions.

4. As a part of the writing procedure many of these programed chapters have been tried out with classes of inservice teachers and classes of preservice teachers. These have not been experimental situations with control groups, etc. The purpose was to find possible errors and misstatements and to discover whether there were sequences of frames in which the programs are inadequate for efficient learning. A number of revisions have been made as a result of these tryouts.

In most of the chapters of the program ideas and concepts are developed; however, Chapters 7, 10, and 11 are intended to develop familiarity with the mechanics of the arithmetical operations previously considered.

HOW TO USE THIS BOOK

This book consists of a large number of questions interspersed with brief expository passages. The questions are enclosed in boxes to separate them from the expository material.

There are two types of questions. The first type consists of a statement of which a part is missing. The location of the missing part is indicated by an underlined blank space. You are to fill in the missing part. The correct answer will be found immediately to the right and outside the box.

The second type consists of a question together with a number of possible answers. You are to indicate which answers are correct. Inside a smaller box each of the possible answers is discussed so you will know if your choices were correct and you will know why you were wrong if you made an incorrect choice.

You should use a sheet of paper to cover up the correct answer in the book until after you have made your response. Then move the sheet of paper just enough to check the answer to that question without exposing the answers to later questions.

You may record your response either in the book or on a separate sheet of paper. In any case you will wish to have scratch paper available to do necessary figuring as you go along.

8

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CHAPTER 1

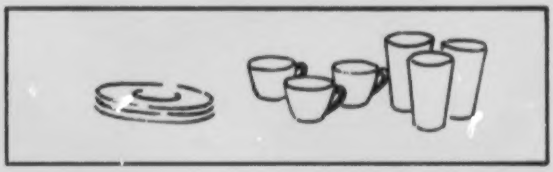
PRE-NUMBER IDEAS

The correct communication of ideas requires an agreement on the meaning of words and symbols. One of the primary purposes of this program is to acquaint the teacher of elementary mathematics with correct mathematical ideas and concepts appropriate to the language of and consistent with more advanced mathematics.

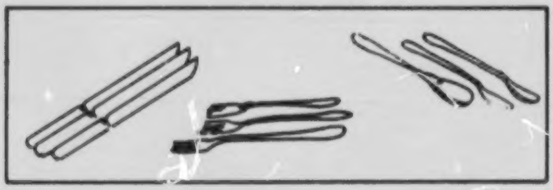
1-1. Sets

The idea of "set" or collection underlies much of the development of mathematics.

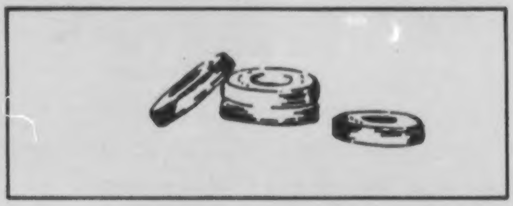
1 The picture below represents a ______ of dishes. set



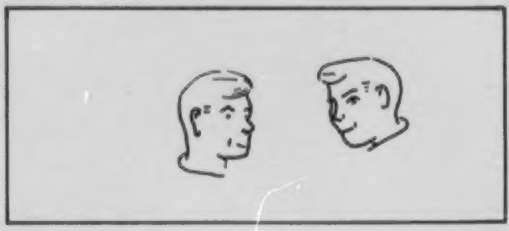
2 This picture represents a ______ of silverware. set



3 This picture represents a ______ of tires. set



4 This picture represents a ______ of twins. set



5 A flock of geese is a ______ of geese. set

6 The ______ of pupils in the third grade has children as members. set

Mathematicians have selected the word set to refer to any collection of objects or ideas. The objects which belong to a set are called elements or members of the set.

7 Jim, Ann, Mary, Alice and Bill are ______ of the set of pupils in the front row. members or elements

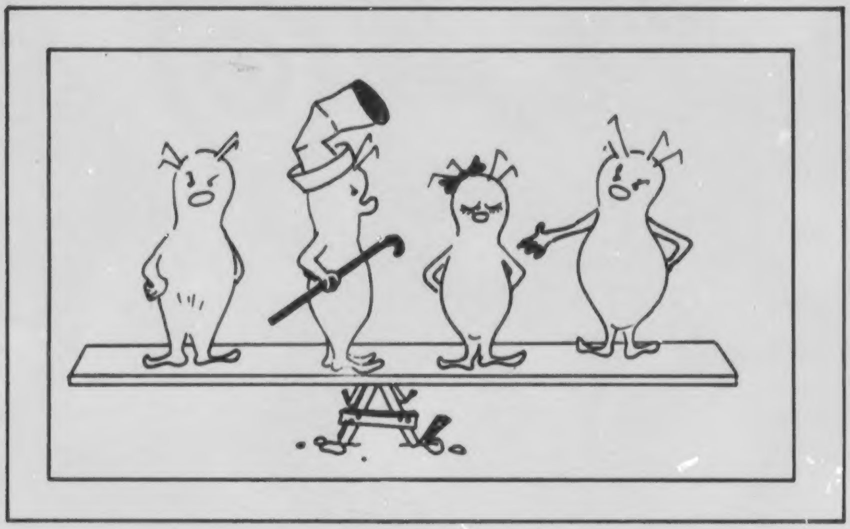


Figure 1.1

8	In Figure 1.1 above are some drawings of things called "gismos." The gismos represent a ______ of things.	set
9	The gismo carrying a cane is a ______ of or an ______ of the set of gismos.	member element
10	If we name the set of gismos with the upper case letter G, then the gismo carrying a cane is a ______ of G.	member
11	A gismo wearing shoes is not an element of the set ______.	G

The gismos in Figure 1.1 are named Bert, Gert, Ace and Ike in the left to right order shown.

12 G = {Gert, Ace, Ike, Bert} represents the set of gismos. Bert is: (Check appropriate answers.)

☐ (a) a member of G

☐ (b) an element of G

☐ (c) a five letter word

☐ (d) a member or element of G

Answers (a), (b) and (d) are correct. Note that a specific order is not implied. Answer (c) is obviously incorrect.

The relation "equal" is used to show that objects are the same and only the names may be changed.

13	It follows from the preceding statement that a rearrangement of the elements of a set results in an _______ set.	equal
14	Two sets are equal if they have the same _______ regardless of how they are specified.	elements
15	Equality is designated by the symbol = . If A and B denote sets, then A = B means the elements of A are the same as the elements of _______.	B

The braces { } are used to enclose a listing of the elements of a set and are read "the set." Occasionally braces are used to enclose a word description of a set.

16	Using the notation with braces write the set of vowels of the English alphabet. _______	{a,e,i,o,u}

17	If the upper case letter V is used to name the set of vowels, then $\{a,e,i,o,u\}$ = ______.	V
18	In Frame 17, u is an element of ______.	V
19	Mathematical symbols may be used to write the statement, "u is an element of V." This is written as $u \in V$. The symbol $\in$ means ______.	is an element of or is a member of
20	The denial that an element is a member of a set can be symbolized by a slash across the membership symbol. Thus, $m \notin V$ is read ______.	"m is not an element of V."

21 Check the correct mathematical statement or statements for the sentence, "Bert, the gismo, is a member of the set denoted by G."

☐(a) Bert $\in$ G

☐(b) G $\in$ Bert

☐(c) r $\in$ G

☐(d) Bert $\in$ $\{$Ace, Ike, Bert, Gert$\}$

21(a) Correct. 21(d) also is correct.

21(b) Incorrect. G is a name for the set, not an element of the set.

21(c) This would be correct if r were another name for Bert. It is an incorrect response, however.

21(d) Correct. 21(a) also is correct.

Just as upper case letters customarily are used to name a set, lower case letters often are used to name an element or member of a set.

22 If we use the first letter of each gismo's name, written in lower case, to represent the gismo, then among the gismos

$$G = \{\text{Bert, Gert, Ace, Ike}\}$$

$a \in G$ is a statement meaning ______.

Ace is an element of the set of gismos.

23 Using a symbolic statement similar to that in Frame 18, state "Gert is a member of the set of gismos." ______

$g \in G$

24 Let $A = \{a,b,c,d,e, \ldots ,y,z\}$. The symbol ... is called the ellipsis and means "and so on." Which of the following statements are correct?

☐(a) $d \notin A$ ☐(c) $m \in A$

☐(b) $y \notin a$ ☐(d) $A \in a$

24(a) Incorrect, d is an element of A.

24(b) Incorrect. This is a misuse of symbols. An upper case letter is used to designate a set and "a" is a lower case letter.

24(c) Correct. Although "m" is not specifically listed, it is implied by the ellipsis.

24(d) Incorrect. $a \in A$, however.

25 Let B = the set of presidents of the United States. Then, George Washington $\in$ ______.

B

26 Al Capp ______ B.

$\notin$

27 Let M = the set of states of the United States

N = the set of common fractions.

Then, $\frac{1}{2} \in$ ______.

N

28 Texas $\in$ ______.

M

29	Alaska ______ N.	$\notin$
30	Suppose $W = \{0,1,2,3, \ldots\}$. Then, 7 ______ W.	$\in$
31	And, $\frac{4}{5}$ ______ W.	$\notin$

The reader may have observed that some of the sets considered have only a few members, others have many members, and some have no end to an attempted listing of elements. If every element can be listed, we say that the set is finite. If a set is not finite, it is said to be infinite.

32	G, the set of gismos, has ______________________. (several members, no members, one member)	several members
33	The set of letters of the alphabet is ______________. (infinite, empty, finite)	finite
34	$W = \{0,1,2,3,4, \ldots\}$ is ______________. (infinite, finite, empty)	infinite

A set may have no members at all. If a set has no members, it is named the empty set. Two examples of the empty set are

{mail carried by the pony express in 1963}
{jet planes that existed in 1963 B.C.} .

The convention for using braces in set notation also applies to the empty set. It is designated by { }. The empty space between the braces indicates that there is no member of the empty set. Any example of the empty set has the same members as any other example of the empty set because none of them has any member. This is why we say the empty set; there is only one such set.

35	A symbol used for the empty set is ______.	{ }
36	In Figure 1-1, the set of gismos riding horses is ______. (finite, empty, infinite)	finite and empty
37	R = {apple, elephant, the color red, algebra}. These elements are related by the common characteristic of belonging to ______.	set R
38	The set of letters used to spell "are" is ______.	{a,r,e}
39	The set of letters used to spell "area" is ______.	{a,r,e}
40	The set of letters used to spell "rare" is ______. (It usually is desirable not to repeat the same element in a set.)	{a,r,e}
41	The set of letters used to spell "Oklahoma" is ______.	{O,k,l,a,h,m}

1-2. Chapter Summary

The primary concern of Chapter 1 is an introduction to the language of sets. The idea of a set as a well-defined collection of elements or members is fundamental to the development of this program.

The reader should adopt the special symbols { }, ∈, ∉ and the appropriate use of letters in naming sets. Sets may be made up of abstract ideas, concrete objects, or a combination of these.

At this point, the reader should have the understanding that a set may be finite or infinite. A finite set may be empty. A particular order of listing the elements of a set is not implied by set notation. (Sometimes, for convenience, we do impose an order on a particular set.)

We will arbitrarily agree never to list the same element twice in a listing. However, we do not eliminate the possibility of having two or more elements which look alike. For example, consider the set of prime factors of 8. This set is denoted by {2,2,2} and will be considered in a later chapter.

CHAPTER 2

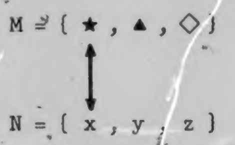
WHOLE NUMBERS

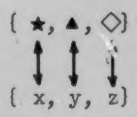
Man's need for counting and ordering collections motivated the creation of number concepts. Sets, then, are pre-number ideas and some of the language, relations and operations of sets contribute much toward clarification and unification of numerical concepts.

2-1. Matching Sets

Before the creation of numbers man could have kept a record of a flock of sheep by pairing each animal with a pebble. He also could have paired each animal with a mark in the sand or with a finger on his hand.

1 In the following example, ★ is paired with x by use of a double arrow. Use double arrows to complete the pairing of the elements of M with the elements of N.

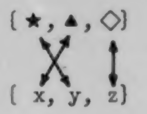




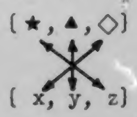
2 Use the sets in Frame 1 to form a different pairing of the members.

M = { ★ , ▲ , ◇}

N = { x , y , z }

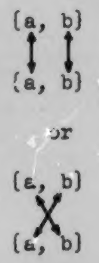


or



There are three other pairings.

3	Try to pair the members of F with those of E. $F = \{\alpha, \beta, \gamma, \pi\}$ $E = \{a, b, c\}$	$\begin{array}{cccc} \{\alpha, & \beta, & \gamma, & \pi\} \\ \updownarrow & \updownarrow & \updownarrow & \\ \{a, & b, & c\} & \end{array}$ Since E is exhausted, we cannot pair each element in F with a correspondent in E.
4	When we exhause the elements of both sets in a pairing, we say that the sets match. Show that one of the sets in Frame 3 matches with N of Frame 1. $N = \{x, y, z\}$ ____ = ____	$N = \{x, y, z\}$ $E = \{a, b, c\}$
5	Given $P = \{a, b, c\}$ $Q = \{▲, ◇, ○, ★\}$ $R = \{⬭, \beta, y\}$ $S = \{$Mary, Dick, Bill$\}$ Which of these sets match? ____	P and R, P and S, R and S
6	If C and D are matched sets and the elements of D are rearranged, then the sets ______ match. (will, may not)	will
7	A one-to-one correspondence between the elements of two sets is a pairing which simultaneously exhausts both sets. Thus, matching sets form a ______ correspondence.	one-to-one

8	Consider M = {a, b} N = {r, s} G = {★, □} Show that M matches M. M = {a, b} M = {a, b}	
9	Since M matches N, then N ______ M.	matches
10	Since M matches N and N matches G, then M ______ G.	matches

Frames 8, 9, and 10 emphasize three important properties of matched sets which mathematicians refer to as the reflexive property, the symmetric property, and the transitive property, respectively. The transitive property illustrated in Frame 10 is especially useful. If A matches B and B matches C, then we know that A matches C without performing a pairing of the elements.

11 Given the following sets:

$D = \{m, n, o\}$ $F = \{u, v, w\}$

$E = \{x, y\}$ $G = \{t\}$

Check the response which correctly relates the sets.

☐ (a) D matches E ☐ (c) F matches D

☐ (b) E matches G ☐ (d) D matches F

11(a) D has more elements than E. This response is incorrect, since any pairing of elements of D and E exhausts the elements of E before the elements of D are exhausted.

11(b) G has fewer elements than E. This response is incorrect, since any pairing of elements of E and G exhausts the elements of G before the elements of E are exhausted.

11(c) Since any pairing of the elements of F with the elements of D exhausts both sets at the same time, this response is correct. 11(d) also is correct.

11(d) Since any pairing of the elements of D with the elements of F exhausts both sets at the same time, this response is correct. 11(c) also is correct.

12 Consider the sets

$P = \{a, b, c\}$ and

$Q = \{★, ○, ◇, ▲\}$.

We find Q has more elements than P, since on pairing the elements of P and Q the elements of Q are not ______. In this case, we say that Q is more than P.

exhausted

13 Using the sets of Frame 12, we see P has fewer elements than Q, and we say that P is ______ than Q.

less

14 Let A and B denote sets. In a pairing of the elements of A and B there are the following possibilities:

(1) The elements of B are exhausted before the elements of A; in this case, we say that A has more elements than B.

(2) The elements of A are exhausted before the elements of B; in this case, we say that A has fewer elements than B.

(3) The elements of A and B are exhausted simultaneously; in this case, we say that A ______ B.

matches

15 Given the sets

$A = \{⬬, \diamond, \blacktriangle, \infty\}$

$B = \{u, v\}$

$C = \{\bigstar, ⬬\}$

Check each response which correctly relates the sets.

☐ (a) There is a one-to-one correspondence between the elements of A and B.

☐ (b) C matches B.

☐ (c) C is less than A.

☐ (d) Any pairing of the elements of B with the elements of C is a one-to-one correspondence.

15(a) Incorrect. There is no one-to-one correspondence between A and B.

15(b) Correct. 15(c) and 15(d) also are correct.

15(c) Since a pairing exhausts C before A, this response is correct. 15(b) and 15(d) also are correct.

15(d) Correct. 15(b) and 15(c) also are correct.

16 Check all correct responses for the given sets:

$A = \{x, y, z, w\}$

$B = \{\bigstar, \Delta, \pi, \infty\}$

$C = \{r, s, x, y\}$.

(a) Since A matches B and B matches C, then A matches C.

(b) C matches C.

(c) $\bigstar \in B$

(d) $y \in B$

(e) There is a one-to-one correspondence between the members of C and the members of B.

16(a) Correct. This is an example of the transitive property of a relation. See also 16(b), 16(c), 16(e).

16(b) Correct. This is an example of the reflexive property of a relation. See also 16(a), 16(c), 16(e).

16(c) Correct. $\bigstar$ is a member of B.

16(d) Incorrect. $y \notin B$. However, $y \in C$ and $y \in A$.

16(e) Correct. Since C matches B and B matches C, then we have an example of the symmetric property of a relation.

17 If K denotes the set of letters used to spell the word "attract," then

K = ______.

$\{a, t, r, c\}$

18 If H denotes the set of letters used to spell the word "cataract," then

H = ______.

$\{c, a, t, r\}$

19	Using the sets in Frames 16 and 17, we say that K ______ H and H matches ______.	matches; K

Frames 17, 18 and 19 should lead the reader to conclude that K matches H since there exists a one-to-one correspondence between the members of K and the members of H. Also, K and H are different representations of the same set and any set matches itself.

20	Consider R = {a, b, c, d} S = {u, v, w, x, y} X = {○, ★, ◇} Any pairing of the elements of X with the elements of R exhausts X before R. Hence, X is ______ than S.	less
21	R is ______ than S.	less
22	Since X is less than R and R is less than S, then X is ______ than S.	less
23	S is more than R and R is more than X, therefore S is ______ than X.	more

Frames 20 through 23 are intended to illustrate the transitive property of the "more than" and "less than" relations for sets.

24 If R is more than S and S is more than T, then: (Check all correct responses.)

☐ (a) R matches T.

☐ (b) R is more than T.

☐ (c) R is less than T.

24(a) Incorrect. Review Frames 20 - 23.

24(b) Correct. Continue to next frame.

24(c) Incorrect. Review Frames 20 - 23.

25 If M is more than N and N is less than P, then: (Check one.)

☐ (a) M is more than P.

☐ (b) M is less than P.

☐ (c) M matches P.

☐ (d) No correct conclusion can be drawn from the given conditions.

25(a) Incorrect. Consider M = {a, b, c}, N = {x, y}, P = {r, s, t, v}.

25(b) Incorrect. Consider M = {a, b, c, d}, N = {x, y}, P = {r, s, t}.

25(c) Incorrect. Consider the sets in either 25(a) or 25(b), both of which satisfy the given conditions, M is more than N and N is less than P.

25(d) Correct. While exactly one of the relations (a), (b), (c) must be true, we do not have enough information to determine which one.

2-2. Number

Consider the following sets for Frames 26 - 37.

A = {▲, ○, ★, ◇} C = {β, α, γ, π}

B = {a, b, c, d} D = {*, +, ◇}

26	A matches B and B ______ C.	matches
27	A ______ C.	matches
28	Exhibit a set F distinct from A, B and C such that F matches A, B and C. F = ______.	For example, {z, y, t, v}
29	F ______ A.	matches

30	F ______ B.	matches
31	F ______ match C. (does, does not)	does
32	F ______ match D. (does, does not)	does not
33	We call the common property of the matching sets A, B, C, F and all other sets which match these the *number property* of the sets. In this example, *fourness* is the common ______ of these matching sets.	property
34	If S denotes any finite set, then the ______ property of S may be represented by the symbol N(S).	number
35	If A = {▲, ○, ★, ◇}, then N(A) = ______.	4
36	Since B = {a, b, c, d} matches A, then N(B) = N(A) = ______.	4
37	If D = {*, +, ◇}, then N(D) = ______.	3

38 If N(R) = 4, N(S) = 6, N(F) = 3, and T is any set which matches S, then: (Check all correct answers.)

☐ (a) N(T) = 4 ☐ (b) N(T) = 6 ☐ (c) N(T) = 3

38(a) Incorrect. T matches S, but N(T) = N(S) = 6.

38(b) Correct. T matches S and N(T) = N(S) = 6.

38(c) Incorrect. T does not match F.

The number property associated with a set is called the *cardinal number* of the set.

39	If $A = \{m, u, d\}$, then the cardinal number of A is ______.	3
40	N(A) is a symbol for the ______ number of A.	cardinal
41	The cardinal number of the empty set is ______.	0 or zero
42	If B is more than A, then N(B) is said to be greater than ______.	N(A)
43	$N(B) > N(A)$ is a symbolic way of stating the relation between the number properties of the sets in Frame 42. Hence, the symbol ______ means "is greater than."	>
44	If B is less than A, we reverse the symbol in Frame 43 and write N(B) ______ N(A).	<

45 Consider G = {★, ◇, ○}

H = {t, o, p, s}

$I = \{\alpha, \beta, \gamma\}$

Which of the following are correct:

☐ (a) G = I ☐ (b) $N(H) > N(I)$ ☐ (c) $N(G) < N(H)$

45(a) Incorrect. These sets are not identical.

45(b) Correct. 45(c) also is correct.

45(c) Correct. 45(b) also is correct.

The common number property (or cardinal number) shared by all sets which match has been our early consideration. The collection of all cardinal numbers assigned to finite sets is called the set of non-zero whole numbers. The whole number zero is the number property (or cardinal number) of the empty set.

The relations < and > are used to impose an order on the set of whole numbers. This order, in turn, simplifies the problem of indicating a listing of these numbers. Thus, if we consider a finite set of whole numbers such as {6, 0, 2, 1, 4} and order its elements in a row such that each is less than the one to its right, we obtain {0, 1, 2, 4, 6}.

Order the elements of each of the following sets using the left to right ordering of <.

46	{4, 2, 3}	______	{2, 3, 4}
47	{6, 2, 3, 5, 4, 0, 1}	______	{0,1,2,3,4,5,6}
48	{8, 2, 6, 4, 0}	______	{0,2,4,6,8}

This procedure motivates us to use the order on the set of whole numbers as follows:

$$W = \{0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots\} .$$

49	If a pairing of the elements of A with the elements of B exhausts B with exactly one element of A left over, then N(A) is called the successor of ______.	N(B)
50	If N(B) = 5 and N(A) = 6, then N(A) is the ______ of N(B).	successor
51	If W, the set of whole numbers, is ordered so that between any pair of elements from left to right we correctly could write <, then W = ______.	{0,1,2,3,4,5,...}
52	If W is ordered as in Frame 47, then every element is to the left of its ______.	successor

53	Delete the element *zero* from the set W. The set of elements remaining is called the set of *counting numbers*. Thus, C = ______.	{1,2,3,4,5,6, ...}
54	An obvious, but frequently overlooked fact should be noted about C, the set of counting numbers. If we impose the same order used in Frame 53, and select any finite set D beginning with *one* and include all successive numbers up to some number *n*, then N(D) = ______.	n
55	N({1, 2, 3, 4, ..., 87, 88, 89}) = ______ .	89
56	Let R denote any set. To obtain the *count* of R we match the elements of R with the elements of a set D, as D is described in Frame 54. Then, N(R) = N( ).	N(D)
57	If R = {b, a, d, c, f, e}, then D = {1, 2, 3, 4, 5, 6} and N(R) = N(D) = ______.	6
58	The order relation on the set of counting numbers is a matter of convenience. N({3, 1, 2, 9, 4, 7, 5, 6, 8}) = N({1, 2, 3, ..., 8, 9}) = ______.	9
59	Order the members of S = {5, 7, 1, 3, 6, 4, 2} using the relation > from left to right. S = ______.	{7,6,5,4,3,2,1}
60	Order the set in Frame 59 using the relation < from left to right. S = ______.	{1,2,3,4,5,6,7}

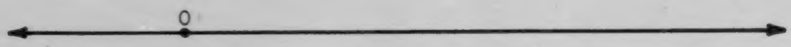
61	The cardinal number of the set in Frame 60 is ______.	$N(S) = 7$
62	Obtaining the count of a set is called counting. Counting gives us the ______ property or cardinal number of a set.	number
63	The number sentence for the statement, "Six is greater than two," is ______.	$6 > 2$
64	Let $N(A) = 4$ and $N(B) = 7$. Since $N(A) < N(B)$, we may write the number sentence ______.	$4 < 7$

2-3. Number Sentences

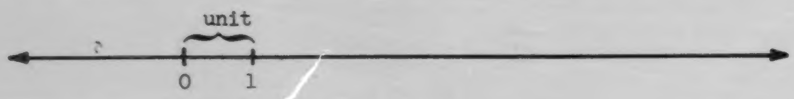
$6 > 4$, $3 < 5$, $4 = 2$ are examples of what we designate as number sentences. Notice that a number sentence is not necessarily true.

A useful device is the pairing of numbers with points on a line. If we represent a line by the sketch

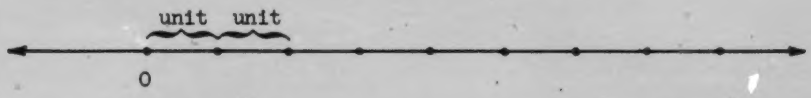
then we may establish a pairing of the whole with points on the line by selecting an arbitrary point and pairing it with 0.



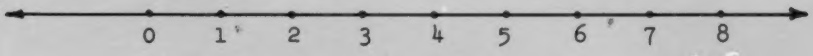
Then we select an arbitrary line segment for a unit.



By common convention, this unit segment is used to space successive points to the right of the point labelled 0.



We then use the set of whole numbers ordered by < and pair these points from left to right with whole numbers.



65	A line used in this manner is called a number line. The point paired with ______ is to the left of every point paired with a counting number.	0
66	Since 10 ______ 7, we expect the point paired with 10 to be to the ______ (left, right) of the point paired with 7.	> right
67	If the point paired with N(C) is to the left of the point paired with N(D), then N(C) ______ N(D).	<

CHAPTER 3

NAMES FOR NUMBERS

3-1. Introduction

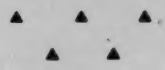
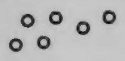
We have introduced the concept of number as an idea and as a symbol associated with collections of matched sets. We have used the phrase "number property of a set" as the identification of the number. By this means we have been able to compare numbers as more than or less than, have been able to order numbers, and have written sets of ordered numbers such as {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} and {9, 8, 7, 6, 5, 4, 3, 2, 1, 0}. Numbers also have been associated with points on a line making a number line.

Now we are interested in systems of writing numbers or systems of numeration. In general we need ways of writing or talking about numbers and in so doing we use the phrase "names for numbers."

3-2. The Beginnings of Numeration

From primitive times "names for numbers" have been both marks or symbols and words. We know little about the primitive man's idea of number. We do know that he tied knots in a rope or collected pebbles and used the idea of one-to-one correspondence between matched sets to indicate the number of fighting men in his tribe, and so forth. It is probable that he counted using his fingers. Today, some primitive tribes use the word for "hand" to indicate five. Since he had ten fingers, the American Indian used "one Indian" to indicate the number concept "ten."

Primitive man made marks such as ///// on the walls of the caves in which he lived to indicate numbers. Probably before this he made marks in the sand or other soil for the same purpose. When this occurred, he was writing the first names for numbers. Today we call these names for numbers numerals.

1	We already know that number is an abstract idea and we speak of "the number property of a collection of matched sets." Primitive man used the symbol /// as a numeral for the number ______. A numeral is a name for a number; it represents the number.	3 or three
2	The numeral we use for the set of triangles is ______. 	5 or five
3	The Roman numeral for the set of triangles in Frame 2 is ______.	V
4	A primitive man's numeral for the set of coconuts  would have been ______.	////// or knots in a string, etc.
5	The Roman ______ for the set in Frame 4 is VI.	numeral
6	In mathematics we use the symbol = to indicate that two symbols are names for the same thing. From Frames 4 and 5 we may write ______ = ______.	////// = VI
7	Let A and B name sets. Then A = B means that all the elements of A also are elements of B and all the elements of B also are ______ of A.	elements
8	The Roman numeral for 18 is ______.	XVIII
9	Using the symbol = and a Roman numeral we may say 18 ______.	18 = XVIII

10	We may write 9 = IX because 9 and IX are ______ for the same number.	names
11	We say 3 + 4 = 8 - 1 because 3 + 4 and 8 - 1 ______ the same number, that is 7.	name or represent
12	Three other names for the number seven are ______, ______, ______.	/////// 5 + 2 VII

In this sub-program we have been interested in symbols as names for numbers. Primitive man used simple symbols and had only a few of them. Roman numerals, with which all of us are familiar, are more complicated. As we shall see, the Romans had ways of extending the idea of numeration to include much larger numbers. We shall be interested in what some of the ancient civilizations did about these extensions to fit their more complicated needs. We call these numeration systems.

3-3. Numeration Systems

13	Using Roman numerals write the numeral associated with A = {John, Mary, Bill}. ______	III
14	The Roman numeral associated with B = {a, c, b, e, f, g} is ______.	VI
15	VIII is a ______ numeral.	Roman
16	The Roman numerals for the first three counting numbers are ______.	I, II, III
17	Roman numerals are marks or symbols for naming ______.	numbers

18	The Roman numeral for the number ten is ______. This is a new symbol not used before in this chapter.	X
19	The numeration system we use today is called the Hindu-Arabic numeration system. In this system the symbol or name for the number ten is ______.	10
20	The name for the number thirteen in the Roman system is ______; but in the Hindu-Arabic system it is ______.	XIII 13
21	The Egyptians wrote their basic numerals up to ten by the use of a vertical stroke with repeated strokes. The Egyptian numeral for seven was written ______.	/////// or //// ///
22	The Egyptians wrote nine as ______.	///// //// or /////////
23	The Egyptians also grouped by tens and invented the symbol ∩ (a heel-bone) for ten. Using this symbol they wrote 13 as ______.	∩///
24	45 was written in Egyptian numerals as ______.	∩∩∩∩/////
25	However, 45 was written in Roman numerals as ______.	XLV
26	For 100 the Egyptians used the symbol 𓍢 (a coiled rope) and the Romans used the letter C. Hence, 100 = ______ = ______.	100 = 𓍢 = C
27	223 was written as ______ by the Romans and as ______ by the Egyptians.	CCXXIII 𓍢𓍢∩∩///

28 The Egyptians, the Romans and all ancient peoples were forced to invent new symbols since a single symbol was not repeated more than nine times. Thus, in Egypt the following different symbols were used for 1, 10, 100, 1000, et cetera:

1	𓏺	the vertical stroke
10	𓎆	the heel-bone
100	𓍢	the coiled rope
1,000	𓆼	the lotus flower
10,000	𓂭	the pointing finger
100,000	𓆐	fish or polliwog
1,000,000	𓁨	astonished man

These symbols were sufficient for their needs. The number 2321 was written by the Egyptians as ______.

𓆼𓆼𓍢𓍢𓎆𓎆𓏺

The Egyptians had no symbol for the number zero.

29 The Romans ______ have a symbol for
(did, did not)
the number zero.

did not

In this sub-program we have introduced the idea of a numeration system and have used the familiar Roman numerals. Mostly, however, we have used the Egyptian numeration system because it is more primitive and less complicated. Both systems, as well as the Hindu-Arabic system we use, made groupings on the basis of ten. Because man has ten fingers, most of mankind's numeration systems grouped on the base ten. We call all such systems base ten systems of numeration.

Thus, the Egyptians used single strokes to represent all numbers up to ten; for example, // for 2, //// for 4. However, instead of ten strokes they wrote the heel-bone symbol 𓎆 for ten. This symbol could be repeated up to nine times and with these repetitions all of the numbers up to 999 could be written. For the number 1000 another symbol , a drawing of a coiled rope used in surveying or perhaps the end of a scroll, was used. If the stroke is included, a total of seven symbols were invented and used. These were sufficient for the needs of the Egyptians.

Neither the Romans nor the Egyptians had a symbol for the number property of the empty set, which we write as 0. However, some of the ancient systems did have such a symbol. The Hindu-Arabic system did not use 0 until about 900 A.D.

3-4. Zero and the Decimal System

30	The number property of the empty set is represented by ______.	0 or zero
31	In the Hindu-Arabic numeration system the first ten numbers are not represented by strokes or the repetition of a single symbol. The set of numerals for the first ten numbers is ______. These are called the basic numerals.	{0,1,2,3,4,5, 6,7,8,9}
32	To write another numeral for ten in the Hindu-Arabic numeration system we use 0 and 1 in the following manner: ______.	10
33	To write another numeral for thirteen we use 1 and 3 in the following manner: ______.	13
34	The number seventy-two is written ______.	72
35	Another numeral for sixty-nine is ______.	69
36	To write a numeral in the hundreds we use a basic numeral in the third place from the right. For example, the numeral for eight hundred sixty-four is ______.	864

37	Another numeral for nine hundred six is ______.	906
38	Since in Frame 37 there are no tens in the number, we use ______ to fill the second place.	0
39	The numeral 3006 represents the number ______.	three thousand six
40	The Egyptians and the Romans ______ (had, had no) need for a 0.	had no

Summary

This chapter has introduced the idea of names for numbers. We call these numerals. We have always spoken of Roman numerals. The concept of number is an abstract idea, the number property of a collection of matched sets. A number has many names, those used by other peoples as well as symbols written in various ways such as ////, IV, 4, and so forth. We know that we also may write 4 as 2 + 2, 5 - 1, $\frac{8}{2}$, 8 + 2, and so forth. Since the symbol = is used to relate different names for the same thing (number), we may write

$$//// = \text{IV} = 4 = 8 + 2 = \frac{8}{2}.$$

We have talked briefly about numeration systems which are ways in which number names or numerals may be written. These are different for different times and peoples. We are familiar with and use both the Roman and the Hindu-Arabic numeration systems. To some extent we considered the ancient Egyptian numeration system primarily to show that the Hindu-Arabic system has many advantages over the ancient systems.

In the next chapter, we consider place value systems in more detail. In this chapter we did show that the use of 0 and the place value principle have certain advantages. For these and other reasons the Hindu-Arabic numeration system is used almost everywhere in the world today.

CHAPTER 4

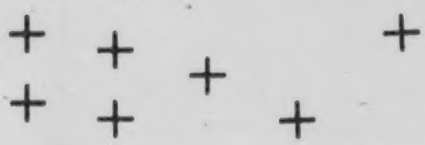
NUMERATION SYSTEMS WITH BASE TEN AND OTHER BASES

4-1. The Meaning of Numerals to the Base Ten

We are concerned in this chapter with the Hindu-Arabic numeration system. This system is commonly known over the world today, but some of its basic properties often are not well known. We will see, before the end of this chapter that the same properties apply to numeration systems to bases other than ten.

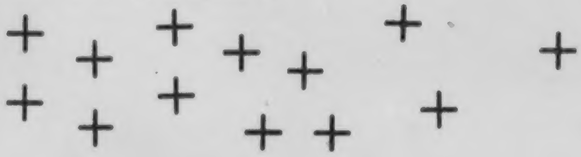
1 If A = {ball, bat, car, boat},
then the number property of A is ______. 4

2 If the elements of a set are represented by crosses as follows:



then the number property of the set is ______. 7

3 In a set represented by



we often rearrange these elements or group them in the following manner:

$\boxed{++++++++++}\quad +++$

and the number property of the set is represented
by the numeral ______. 13

4	In the numeral 13, the 1 stands for or represents ______ objects or elements of the set of crosses.	10
5	In the numeral 13, the 3 represents ______ elements.	3
6	The numeral 0 is the name for a ______ number and indicates the set having no elements.	whole
7	$N(\{\ \}) =$ ____.	0
8	In the numeral 10, the 1 represents one group of ______ objects.	10 or ten
9	In the numeral 15, 1 represents the number property of a set of ______ objects and 5 represents the number property of a set of ______ objects.	10 or ten 5
10	In the numeral 20, 0 represents a set of no elements, but it also serves the purpose of enabling us to write 2 in the ______ place.	second or tens
11	The basic set of numerals or digits used in the Hindu-Arabic system is ______.	{0,1,2,3,4,5, 6,7,8,9}
12	In the numeral 327, 7 represents a property of a set of ______ units, the 2 a set of ______, and the 3 ______.	7 2 tens; a set of 3 hundreds
13	In the numeral 4056, 0 represents a set of hundreds which has no ______.	elements
14	In the numeral 472, 7 represents 7(10) and 4 represents 4(×).	4(10 × 10)

15	In the numeral 5320, 5 represents ____(× ×).	5(10 × 10 × 10)
16	The numeral 73 may be written as 7(10) + ____.	3
17	The numeral 5473 may be written as 5( ) + ( )+ ( ) + ____.	5(10×10×10) + 4(10×10) + 7(10) + 3
18	Since, in this system, the basic set of numerals has ten elements, its base is ____.	ten
19	In a system with the base ten, if the third digit from the right is b, it represents ____( ).	b(10 × 10)
20	If the sixth digit from the right is g, it represents ____( ).	g(10×10×10×10×10)

Summary

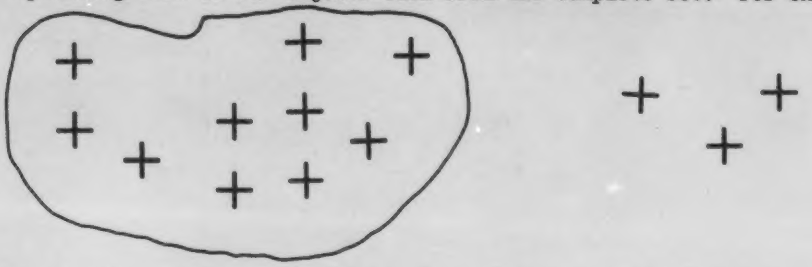
The value represented by a digit or basic numeral of the Hindu-Arabic numeration system depends both on the digit itself and on the place it occupies in the complete numeral. That is, in 173, the digit 7 actually represents 7(10) or 70 and the 1 represents 1(10 × 10) or 1(100) or 100. Since there are ten basic numerals or digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, we say the system has a base ten. In a given numeral such as 7396, the first place on the right tells us how many objects or units are included, that is, 6(1); the second place to the right tells us how many sets of (10) are included, that is, 9(10); the third how many sets of (10 × 10); the fourth how many sets of (10 × 10 × 10). Hence, the number may be written

$$7396 = 7(10 \times 10 \times 10) + 3(10 \times 10) + 9(10) + 6(1).$$

We shall find that an analogous thing occurs when numeration systems to other bases are used.

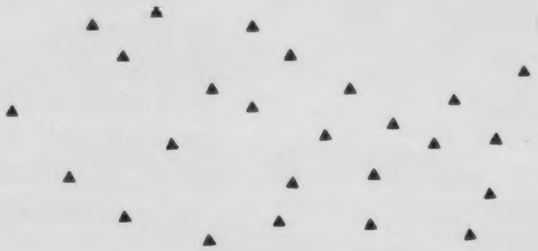
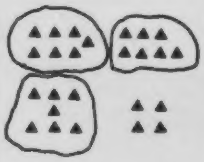
4-2. Numeration Systems to Other Bases

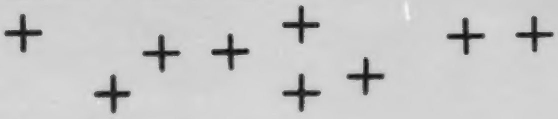
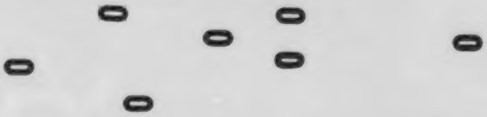
As we have said, the numeration system to the base ten is built up by separating sets of ten objects each from the complete set. For example,



Then by the use of positional notation or place value, we are able to write names for numbers larger than 10 using the same basic digits.

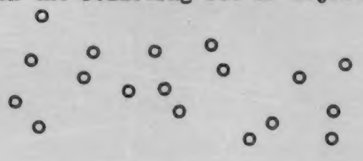
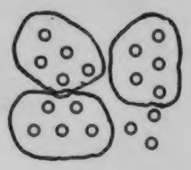
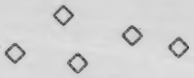
Separating a set of objects into sets smaller or larger than ten may be done similarly and this leads us to consider numeration systems to other bases. We use base seven and base five as illustrations in the following programs. Any other base may be used. Historically, bases of five, twelve, twenty and sixty have been used.

21	Separate the following set of objects into sets of seven elements each. 	 (The arrangement of objects is not unique, but there should be 3 sets each with 7 objects and 4 more objects.)
22	We observe that there are ______ sets with 7 elements in each set and ______ elements left over.	3 4
23	The number of objects in Frame 21 to the base ______ is written 34_{seven}.	seven or 7
24	The 3 in 34_{seven} represents ______ objects.	3(7) or three times seven or 21_{ten}

25	The number of objects in the set of elements represented by the crosses  may be written to the base seven as ______.	12_{seven}
26	The number of objects in  may be written to the base seven as ______.	10_{seven}
27	In the numeral 342_{seven}, the digit 4 means ____(10_{seven}) or $[4(10)]_{seven}$.	4_{seven}
28	A numeration system to the base seven has the following set of seven basic digits: ______	{0,1,2,3,4,5,6}

Note: Although it is not absolutely necessary, we follow the practice of using symbols such as 5_{seven}, 3_{five}, $5_{seven}(10_{seven})$, and the like, each time we refer to a base other than ten. As in Frame 27 above, an expression such as $4_{seven}(10_{seven})$ sometimes will be shortened to $[4(10)]_{seven}$ where the square bracket is used to indicate that all numerals within the brackets are understood to be to the base seven, five, and so on. For emphasis and clarity, we sometimes will use symbols such as 8_{ten}, 22_{ten}, or $[3(10) + 2]_{ten}$. But if no base is indicated, it will be understood to mean base ten.

29	The numeral 32_{seven} is the same as $[3(\underline{\quad\quad}) + 2]_{seven}$.	10
30	10_{seven} $\underset{\text{(is, is not)}}{\underline{\quad\quad\quad\quad}}$ equal to 7_{ten}.	is
31	$[5(10 \times 10)]_{seven}$ is $\underset{\text{(equal, not equal)}}{\underline{\quad\quad\quad\quad}}$ to $[5(7 \times 7)]_{ten}$.	equal
32	$352_{seven} = [3(10 \times 10) + 5(10) + 2]_{seven}$ and $[3(7 \times 7) + 5(7) + 2]_{seven}$ are different $\underline{\quad\quad}$ for the same number.	names
33	$[3(7 \times 7) + 5(7) + 2]_{ten} = \underline{\quad\quad}_{ten}$.	184
34	The numeral 234_{seven} may be written as $\underline{[2(\quad) + 3(\quad) + 4]_{seven}}$.	$[2(10 \times 10) +$ $3(10) + 4]_{seven}$
35	3050_{seven} may be written as $\underline{[3(\quad) + 0(\quad) + (\quad) + \quad]_{seven}}$.	$[3(10\times10\times10) +$ $0(10\times10) + 5(10)$ $+ 0]_{seven}$
	or as $\underline{[\quad(\quad) + 5(\quad)]_{ten}}$.	$[3(7\times7\times7) +$ $5(7)]_{ten}$
	or as $\underline{\quad\quad}_{ten}$.	1064_{ten}
36	The use of 0 in the numeral 3050_{seven} indicates that the set of $(10 \times 10)_{seven}$ is $\underline{\quad\quad}$ and that the set of $\underline{\quad\quad}$ is empty.	empty; ones or units

37	Given the following set of objects:  Separate this set into sets of five.	
38	To the base five, the number of objects in the set in Frame 37 is written ______.	33_{five}
39	In the numeral 33_{five}, the second 3 from the right represents ______ objects.	$[3(10)]_{\text{five}} =$ $3(5)_{\text{ten}}$ or 15_{ten}
40	The number of objects in the set  is written $\underline{\qquad}_{\text{five}}$.	10_{five}
41	$33_{\text{five}} = \underline{[\quad(\quad)+\quad]}_{\text{five}}$ $= \underline{[\quad(\quad)+\quad]}_{\text{ten}}$ $= \underline{\qquad}_{\text{ten}}$	$[3(10)+3]_{\text{five}}$ $[3(5)+3]_{\text{ten}}$ 18_{ten}
42	$243_{\text{five}} = \underline{[\quad(\quad)+\quad(\quad)+\quad]}_{\text{ten}}$ $= \underline{\qquad}_{\text{ten}}$	$[2(5\times5)+4(5)$ $+3]_{\text{ten}}$ 73_{ten}

43	$3333_{five} =$ $\underline{[3(\quad\quad)+\ (\quad\quad)+\ (\quad\quad)+\ \quad]_{five}}$	$[3(10\times10\times10)$ $+ 3(10\times10)$ $+ 3(10)$ $+ 3(1)]_{five}$
44	$434_{five} = \underline{\quad\quad\quad ten}$	$[4(5\times5) + 3(5)$ $+ 4]_{ten}$ $= 119_{ten}$
45	$434_{five} = \underline{\quad\quad\quad seven}$	230_{seven}
46	$262_{seven} + 341_{five} = \underline{\quad\quad\quad ten}$	238_{ten}

Summary

We may summarize this sub-program in which ideas about numeration systems to the base seven and base five have been developed as follows:

In a numeration system to the base seven there is, of course, a set of seven basic digits, {0, 1, 2, 3, 4, 5, 6}, as was stated in Frame 28. A set of seven objects has a number property written as 10_{seven}. To avoid confusion any numeral to the base seven has the base written to the right and below the symbol, for example, 4_{seven}, 236_{seven}. For the sake of brevity, this was not done in the set {0, 1, 2, 3, 4, 5, 6} above or in the response to Frame 28. We do, sometimes, enclose several numerals in brackets and consider all of them to the base seven, for example, $[5(10 \times 10) + 1(10) + 0]_{seven}$. In the numeral 236_{seven}, the second digit from the right is in terms of 10_{seven} and means $[3(10)]_{seven}$. Since 10_{seven} is another name for 7_{ten}, and $3_{seven} = 3_{ten}$, it is possible to write $[3(10)]_{seven} = [3(7)]_{ten}$. Hence, we may write the numeral 5324_{seven} as follows:

$$5324_{seven} = [5(7 \times 7 \times 7) + 3(7 \times 7) + 2(7) + 4]_{ten}$$

Multiplying and adding as indicated gives the number 1880 to the base ten. We do not need to write the word "ten" here since it is understood. Since these are different names for the same number, it is proper to write $5324_{seven} = 1880$. Use of the base five, or any other base, follows the same procedure.

It should be clear by now that with a numeration system to the base seven, the numerals 7, 8 and 9, as well as 17, 27, 38, 83 and so forth, do not exist. We have used the word "seven" to indicate the base, but at all places where the numeral has been used, the numbers were written in the base ten. The numeral 10_{seven}, often read, "one oh, base seven," is the number property of {X, X, X, X, X, X, X} and all sets which match it. The numeral 7_{ten} is also a number property of these sets and hence, $10_{seven} = 7_{ten}$. This gives us the relationship between these two numeration systems and enables us to move from one to the other.

In like manner, with a system to the base five, the numerals 5, 6, 7, 8, 9, 15, 16, 26, 72, and so forth, do not exist. As discussed above, we may write $10_{five} = 5_{ten}$ and use this basic relation to shift from one system to the other.

It will be useful in understanding numbers to other bases to learn to perform the ordinary operations such as addition and multiplication with them. These will be discussed in the next program.

4-3. Addition and Multiplication with Numerals to Base Seven and Base Five

Addition and multiplication will be discussed formally in other chapters, but a brief discussion of these operations with numerals to other bases will prove useful in understanding the basic properties of positional numeration systems.

47 With numerals to the base ten,

$$4_{ten} + 2_{ten} = 6_{ten}.$$

Similarly, with numerals to the base seven,

$$4_{seven} + 2_{seven} = \underline{\quad\quad}_{seven}.$$

6_{seven}

48	With numerals to the base seven, $3_{seven} + 2_{seven}$ represents a number with which we associate the numeral $\underline{(\quad)_{seven}}$.	$(10)_{seven}$
49	We also may write $3_{seven} + 6_{seven}$ as $3_{seven} + (4_{seven} + 2_{seven})$ $= (3_{seven} + 4_{seven}) + 2_{seven}$ $= \underline{(\quad)_{seven} + \quad_{seven}}$ $= \underline{(\quad)_{seven}}$.	$(10)_{seven} + 2_{seven}$ $(12)_{seven}$
50	And $5_{seven} + 5_{seven}$ may be written as $5_{seven} + (2_{seven} + 3_{seven})$ $= \underline{(\quad)_{seven} + \quad_{seven}}$ $= \underline{(\quad)_{seven}}$.	$(10)_{seven} + 3_{seven}$ $(13)_{seven}$
51	$4_{five} + 4_{five}$ $= 4_{five} + (1_{five} + 3_{five})$ $= (4_{five} + 1_{five}) + 3_{five}$ $= \underline{\quad_{five}}$.	13_{five}
52	$3_{five} + 4_{five}$ $= (3_{five} + 2_{five}) + 2_{five}$ $= \underline{\quad_{five}}$.	12_{five}

53 $6_{seven} + 6_{seven}$

$= (\underline{\quad\quad seven} + \underline{\quad\quad seven}) + \underline{\quad\quad seven}$

$= \underline{\quad\quad seven}$

$(6_{seven} + 1_{seven})$
$+ 5_{seven}$
15_{seven}

Note: In Frames 47 - 53, we have tried to show how one adds with simple one-digit numbers to the base seven and to the base five. If you have understood this, you should be able to make an addition table to base seven or base five; perhaps you could make one to other bases. Here is one to base seven:

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

Addition Table to Base Seven

Using this table one is able to find the sum of any pair of one-digit numbers written to the base seven. To become proficient in adding such numbers, one would need to memorize the addition facts, as grade school children must do with numerals to the base ten.

54 **Make an addition table to the base five.**

+	0	1	2	3	4
0	__	__	__	__	__
1	__	__	__	__	__
2	__	__	__	__	__
3	__	__	__	__	__
4	__	__	__	__	__

Addition Table to Base Five

0	1	2	3	4
1	2	3	4	10
2	3	4	10	11
3	4	10	11	12
4	10	11	12	13

55 $2_{five} + 4_{five} = \underline{\qquad five}$

$= \underline{\qquad ten}$.

11_{five}

6_{ten}

56 $6_{seven} + 4_{seven} = \underline{\qquad seven}$.

13_{seven}

57 Add: $13_{seven} + 32_{seven} = \underline{\qquad seven}$.

45_{seven}

58 Add: $14_{five} + 21_{five} = \underline{\qquad five}$.

40_{five}

59 Add: $24_{seven} + 65_{seven} = \underline{\qquad seven}$.

122_{seven}

60 Add: $34_{five} + 23_{five} = \underline{\qquad five}$.

112_{five}

61 Add: $34_{seven} + 23_{seven} = \underline{\qquad seven}$.

60_{seven}

62 Add: $34_{ten} + 23_{ten} = \underline{\qquad ten}$.

57_{ten}

63	Multiply: $(2_{\text{ten}}) \times (3_{\text{ten}})$ $= 3_{\text{ten}} + 3_{\text{ten}}$ $= \underline{\qquad_{\text{ten}}}.$	6_{ten}
64	Multiply: $(2_{\text{seven}}) \times (3_{\text{seven}})$ $= 3_{\text{seven}} + 3_{\text{seven}}$ $= \underline{\qquad_{\text{seven}}}.$	6_{seven}
65	Multiply: $(2_{\text{five}}) \times (3_{\text{five}})$ $= 3_{\text{five}} + 3_{\text{five}}$ $= 3_{\text{five}} + (2_{\text{five}} + 1_{\text{five}})$ $= (3_{\text{five}} + 2_{\text{five}}) + 1_{\text{five}}$ $= 10_{\text{five}} + 1_{\text{five}}$ $= \underline{\qquad_{\text{five}}}.$	11_{five}
66	Multiply: $(4_{\text{seven}}) \times (5_{\text{seven}})$ $= 5_{\text{seven}} + 5_{\text{seven}} + 5_{\text{seven}} + 5_{\text{seven}}$ $= 10_{\text{seven}} + 10_{\text{seven}} + 6_{\text{seven}}$ $= \underline{\qquad_{\text{seven}}}.$	26_{seven}
67	6_{seven} $\underline{\qquad\qquad}$ (is, is not) equal to 6_{ten}.	is
68	6_{seven} $\underline{\qquad\qquad}$ (is, is not) equal to 11_{five}.	is
69	42_{seven} $\underline{\qquad\qquad}$ (is, is not) equal to 28_{ten}.	is not
70	$(4_{\text{five}}) \times (4_{\text{five}}) = \underline{\qquad_{\text{five}}}.$	31_{five}

71 Multiply $(24_{five}) \times (33_{five}) =$ ______$_{five}$.

2002_{five}

72 Construct a multiplication table to base five.

×	0	1	2	3	4
0	___	___	___	___	___
1	___	___	___	___	___
2	___	___	___	___	___
3	___	___	___	___	___
4	___	___	___	___	___

Multiplication Table to Base Five

0	0	0	0	0
0	1	2	3	4
0	2	4	11	13
0	3	11	14	22
0	4	13	22	31

Summary

We have introduced in this sub-program some of the problems involved in adding and multiplying with numerals to bases seven and five. To answer correctly many of these problems one must understand clearly the meaning of place value, and this is one reason for introducing this section here. Similar exercises for the elementary school student would lead to a deeper understanding of the idea of place value. We have exhibited addition and multiplication tables for bases other than ten; these could be memorized. However, memorization is not as important as a grasp of the notion of place value.

4-4. Further Meaning of Numerals to Base Seven

We want to go further into the meaning of the numeration system to the base seven as an illustration of a numeration system to any base B.

73 When we are counting with numbers to the base ten, the next numeral after 9_{ten} is ______.

10 or 10_{ten}

74	Suppose we are counting with numbers to the base seven. The next numeral after 6_{seven} is ______.	10_{seven}
75	The next six numerals to the base seven are ______, ______, ______, ______, ______, ______.	11_{seven} 12_{seven} 13_{seven} 14_{seven} 15_{seven} 16_{seven}
76	16_{seven} means $\underline{[1(\quad) + \quad]_{seven}}$.	$[1(10) + 6]_{seven}$
77	The next numeral after 16_{seven} is ______.	20_{seven}
78	$20_{seven} = \underline{[\quad(\quad) + \quad]_{seven}}$.	$[2(10) + 0]_{seven}$
79	34_{seven} $= [3(10 + 4]_{seven}$ $= \underline{[\quad(\quad) + \quad]_{ten}}$.	$[3(7 + 4]_{ten}$
80	One more than 66_{seven} is $\underline{[\quad(\quad) + \quad + 1]_{seven}}$.	$[6(10)+6+1]_{seven}$
81	$(6 + 1)_{seven} = \underline{\quad\quad}_{seven}$.	10_{seven}

82	$[6(10) + 1(10)]_{seven}$ $= [(6 + 1) \times 10]_{seven}$ $= [10 \times 10]_{seven}$ $= \underline{\qquad\quad}_{seven}$.	100_{seven}
83	The next number after 666_{seven} is ______.	1000_{seven}
84	$1000_{seven} = \underline{[1(\qquad) + \qquad]_{seven}}$.	$[1(10\times10\times10)$ $+ 0]_{seven}$
85	The first five place values of numerals to the base seven are ______, ______, ______, ______, ______.	1_{seven} 10_{seven} $(10 \times 10)_{seven}$ $(10\times10\times10)_{seven}$ $(10\times10\times10\times10)_{seven}$
86	The first two place values from the right to the base B are $\underline{\qquad}_{B}$, and $\underline{\qquad}_{B}$.	1_{B} 10_{B}
87	Five place values to the base B from the right are ______, ______, ______, ______, ______.	1_{B} 10_{B} $(10 \times 10)_{B}$ $(10\times10\times10)_{B}$ $(10\times10\times10\times10)_{B}$

4-5. Summary for the Chapter

The concepts involved in this chapter on numeration systems to the base ten, and also systems to other bases, are necessary to a full understanding of numbers and how we use the symbols for numbers. It is important that we understand the Hindu-Arabic numeration system, because this is the one used widely over the world.

This system of writing the names for numbers involves three basic characteristics:

1. There is a single mark or symbol for each of the 10 basic digits, that is, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
2. There is a symbol 0 for the number property of the empty set.
3. There is a way of writing the names for all numbers greater than 9 by the use of the basic digits and an agreement on the place value or positional value of each. Thus the numeral 4624 means:

$$4624 = 4(10 \times 10 \times 10) + 6(10 \times 10) + 2(10) + 4(1).$$

This numeral also may be written $4624 = 4(10^3) + 6(10^2) + 4$ although we have not used this notation in the above program.

We also discovered in the program that there are numeration systems with other bases. We used those to the base five and base seven as illustrations. It is clear that these systems have the same three basic characteristics as the Hindu-Arabic system. We have found, in working with children and with teachers, that the use of other bases clarifies the basic concepts of the system we use. The lack of familiarity forces the student to understand the basic concepts.

CHAPTER 5

PROPERTIES OF WHOLE NUMBERS
UNDER THE OPERATION OF ADDITION

5-1. The Whole Numbers (Review)

Earlier we developed the idea that a number, such as three, is a common property of a collection of matched sets, such as A = {pencil, bottle, book} and all the sets which match A. We also have said that the numerals 3, III, 2 + 1, 4 - 1, and so forth are different names for the number three. Other ideas were developed which will be included in the following sub-program.

1	The number property of B = {Jane, George, Ellen, Mary} and all sets which match B is ______.	four, 4, IV
2	The symbol 4 is a name for the number four and is called a ______.	numeral
3	Write another set which matches B = {Jane, George, Ellen, Mary}. For example: ______.	{pencil, ink, chalk, pen}
4	List the elements of the set of numerals {4, 3, 7, 2} in ascending order. ______	{2, 3, 4, 7}
5	Write the set of the first ten whole numbers in ascending order. ______	{0,1,2,3,4,5, 6,7,8,9}
6	List the first five ordered whole numbers. ______	1, 2, 3, 4, 5
7	If S is a set, then the number property of S is denoted by ______.	N(S)

8	If C and D denote sets and the members of C and D are in one-to-one correspondence, then N(C) = ______.	N(D)
9	The number property of S = {desk, chair, hat, coat, pen, pencil} is denoted by the ______ 6.	numeral
10	S in Frame 9 has the number property ______.	6 or six or VI

5-2. Sets under the Operation of Union

In this sub-program we use {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, ...} which we speak of as the set of whole numbers. The numeration system is to the base ten.

To develop the basic idea or definition of addition as an operation on a pair of whole numbers we use sets and some properties of sets. The following program will be on sets and the union of sets.

11	Given A = {pencil, bottle, eraser} and B = {desk, chair, pencil, hat, coat, pen}. Write the set whose elements are members of A or B. ______	{pencil, hat, bottle, coat, eraser, pen, desk, chair}
12	Do we need to write "pencil" twice? ______ (yes, no)	no
13	When we put two sets together, as in Frame 11, to form a single ______, we call this the union of the sets.	set

14 The ______ of

S = {chair, desk, book} and
M = {apple, book, chair}

is {chair, desk, book, apple}.
We denote this set by $S \cup M$.

union

15 Write the set which is the union of

B = {tree, rose, pear} and
C = {orange, pear, apple}.

$B \cup C$ = ______.

{tree, rose, pear, apple, orange}

16 Consider B and C, the sets in Frame 15.

N(B) = ______.
N(C) = ______.
$N(B \cup C)$ = ______.

3 or three
3 or three
5 or five

17 Consider the following pairs of sets:

A = {pencil, eraser, bottle} and
F = {desk, pencil, pen, chair, hat, coat}

S = {book, chair, desk} and
M = {apple, chair, book}

B = {tree, pear, rose} and
C = {orange, apple, pear}

Each pair has at least one common ______.

member or element

18 Given

D = {Jane, George, Bill} and
E = {Ellen, Dorothy, Marion, Joe}.

Write the set which is the union of D and E.
$D \cup E$ = ______.

{Jane, George, Bill, Ellen, Dorothy, Joe, Marion}

19	D and E, the sets in Frame 18, do not have a common ______.	element
20	A pair of sets which do not have a ______ element are called disjoint sets.	common
21	{pen, clock, apple, purse} and {chalk, ruler, pencil} are ______ sets.	disjoint
22	The union of $B = \{\text{tree, rose, pear}\}$ and $C = \{\text{pear, apple, orange}\}$ is a set denoted by $B \cup C$. What is the union of $B \cup C$ and $G = \{\text{elm, oak}\}$? We denote this set by $(B \cup C) \cup G$. $(B \cup C) \cup G =$ ______.	{tree,rose,pear, oak,elm,apple, orange}
23	Write $C \cup G$ for the sets in Frame 22. $C \cup G =$ ______.	{apple,oak, orange,elm,pear}
24	Write the set $B \cup (C \cup G)$. $B \cup (C \cup G) =$ ______.	{tree,rose,pear, oak,elm,apple, orange}
25	Is $(B \cup C) \cup G = B \cup (C \cup G)$? ______ (yes, no)	yes
26	Given the following sets: $S = \{\text{book, chair, desk}\}$ $F = \{\text{desk, chair, coat, hat, pencil, pen}\}$ $M = \{\text{book, apple, chair}\}$. Write $S \cup (M \cup F)$ and $(S \cup M) \cup F$. $S \cup (M \cup F) =$ ______. $(S \cup M) \cup F =$ ______.	{chair,pen,desk, book,apple,coat, pencil,hat} {apple,book, chair,desk,pen, coat,pencil,hat}

27	Is $S \cup (M \cup F) = (S \cup M) \cup F$? ______ (yes, no)	yes
28	$N((S \cup M) \cup F)$ = ______.	eight or 8
29	Write $E \cup D$ using E = {Joe, Dorothy, Ellen, Marion} and D = {Jane, Bill, George}. $E \cup D$ = ______.	{Ellen, Bill, George, Joe, Jane, Marion, Dorothy}
30	Is $E \cup D$ the same as $D \cup E$? ______ (yes, no)	yes
31	Since $E \cup D = D \cup E$, we say that D and E are commutative under the operation ______.	union
32	$N(E \cup D)$ = ______.	7
33.	$N(D \cup E)$ = ______.	7

Summary

In this sub-program we have defined the union (sometimes called the join) of two sets A and B to be the set of distinct elements of the two sets. The union of A and B is written as $A \cup B$. The order in which the elements are written makes no difference, and although a particular element may appear in both sets, it appears only once in the union. Since $B \cup A$ is the same as $A \cup B$, we may say $A \cup B = B \cup A$ for any pair of sets A and B. This relation is called the commutative property of sets under the operation of union.

We also recalled that any set S has a number property written as N(S). Since $A \cup B$ is a set, its number property is $N(A \cup B)$. Since $A \cup B = B \cup A$, it follows that $N(A \cup B) = N(B \cup A)$.

The operation of union (or join) has meaning only when applied to a pair of sets. When three or more sets are involved, our procedure in finding the union allows us to work with only two sets at a time. This is no real difficulty, however, since we know how to find the union $A \cup B$ of A and B (or of any two sets). Then we can find the union of $A \cup B$ and C. We have written this as $(A \cup B) \cup C$. Furthermore, we discovered that $(A \cup B) \cup C = A \cup (B \cup C)$ for all sets A, B and C. This relation is called the *associative property* of sets under the operation of union.

By using the commutative property and the associative property a series of three or more sets under the operation of union may be rearranged in any desired sequence, thus, $(A \cup B) \cup C = (B \cup A) \cup C = B \cup (A \cup C)$, and so forth.

The union of three or more sets is a set. Therefore we may think of the number property of the union of three or more sets. And we write $N((A \cup B) \cup C) = N(A \cup (B \cup C))$ as a result of the associative property of sets under the operation of union.

5-3. Disjoint Sets, Whole Numbers and the Operation of Addition

We may use disjoint sets and the number property of sets to define or give meaning to the binary operation of *addition* in the set of whole numbers. Some of the frames in the following sub-program will be for the purpose of recalling and reinforcing the ideas previously developed regarding disjoint sets, the union of sets, and the number property of sets.

34	Two sets which have no common elements are ______ sets.	disjoint
35	Let A and B denote sets. The ______ of A and B is the set which has as its elements those elements which belong to A or to B or to both A and B.	union
36	Since two disjoint sets contain no common elements, their union contains all the ______ in each of the sets.	elements
37	The number property of $B = \{\text{oak, elm, hickory, walnut, pecan}\}$ is represented by $N(B) =$ ______.	5

38	A numeral for the number property of C = {peach, pear, apple} is ______.	3
39	Using B in Frame 37 and C in Frame 38, $B \cup C$ = ______.	{oak,apple, pecan,elm, hickory,walnut, peach,pear}
40	The number property of $B \cup C$ is $N(B \cup C)$ = ______.	eight or 8
41	N(B) = ______ N(C) = ______ $N(B \cup C)$ = ______.	five three eight
42	Given the numbers 2 and 3. Choose a set D having 2 members and a disjoint set E having 3 members. D = ______ E = ______	{paper,pencil} {book,pen,ink}
43	Write the union of D and E, the sets you chose in Frame 42. $D \cup E$ = ______.	{pencil,paper, pen,ink,book}
44	The number property of $D \cup E$ in Frame 43 is $N(D \cup E)$ = ______.	5
45	In Frame 42, we started with the numbers 2 and 3, and found disjoint sets having 2 and 3 members respectively. We then found that the union of these disjoint sets had the number property 5. This suggests the addition sentence ______.	2 + 3 = 5

46	Use this same idea on the disjoint sets B = {oak, elm, hickory, walnut, pecan} C = {peach, pear, apple}. This suggests the addition sentence ______.	$5 + 3 = 8$
47	Thus, if A and B are disjoint sets and if $N(A) = a$ and $N(B) = b$, then $N(A \cup B) =$ ______.	$a + b$
48	Given any two whole numbers e and f, we may find two ______ sets E and F such that $N(E) = e$ and $N(F) = f$.	disjoint
49	If $N(E \cup F) = g$, then ____ + ____ = ____.	$e + f = g$

Summary

In this sub-program we have used the idea of the number properties of two disjoint sets and the number property of their union to provide a way of pairing two numbers and always obtaining a third number. For example, $N(B) = 5$, $N(C) = 3$, and $N(B \cup C) = 8$. We call this association the operation of addition and use the symbol + to denote this binary operation. Thus, we define addition in the set of whole numbers as follows:

Definition: If $N(A) = a$ and $N(B) = b$ where A and B are disjoint sets, then $a + b$ is the number property of $A \cup B$. If $N(A \cup B) = c$, then $a + b = c$.

This is a definition of addition in terms of disjoint sets and the set operation union. If a and b are any two whole numbers, then we can always find disjoint sets A and B such that $N(A) = a$ and $N(B) = b$. The union of these disjoint sets is denoted by $A \cup B$, and $N(A \cup B)$ is some whole number c such that $c = a + b$.

5-4. Properties of Whole Numbers under the Operation of Addition

We recall that earlier in this chapter it was established that there is a binary operation of union on any two sets. From the meaning of this operation applied to sets it is possible to establish that sets are closed, commutative and associative under the binary operation of union. By the use of disjoint sets, we have made a definition for the binary operation of addition for the set of whole numbers.

In the following sub-program, we again use disjoint sets to establish the fact that the set of whole numbers has the commutative, associative, and closure properties under the operation of addition and that there exists an identity element for addition.

50	From Frame 43, what does the commutative property of sets under the operation of union tell us? ______	$D \cup E = E \cup D$
51	What do we know then about the addition sentence in Frame 45? ______	$2 + 3 = 3 + 2$
52	How may we write the addition sentence in Frame 49? ______	$e + f = f + e$
53	From these statements we may say that the set of whole numbers has the ______ property under the operation of addition.	commutative
54	Since the process of associating disjoint sets and their union with whole numbers may be applied to any two ______ numbers, we also say that the set of whole numbers is closed under the operation of addition.	whole

55	Suppose we want to add the three numbers 2, 3 and 5. Consider B = {pencil, paper} C = {peach, pear, apple} D = {oak, elm, hickory, walnut, pecan}. Are these disjoint sets? $\overline{\text{(yes, no)}}$	yes
56	$N(B \cup C) = \underline{(\quad + \quad)}$.	(2 + 3)
57	$N((B \cup C) \cup D) = \underline{(\quad + \quad) + \quad}$.	(2 + 3) + 5
58	$N(B \cup (C \cup D)) = \underline{\quad + (\quad + \quad)}$.	2 + (3 + 5)
59	Since the associative property of sets under the operative union states that $(B \cup C) \cup D = B \cup (C \cup D)$, complete the number sentence $(2 + 3) + 5 =$ ____.	2 + (3 + 5)
60	The sentence $(2 + 3) + 5 = 2 + (3 + 5)$ is an illustration of the ______ property of the set of whole numbers under the operation of addition.	associative
61	Let a, b, c represent any triple of whole numbers. Write the number sentence which states the associative property of the set of whole numbers under addition. ______	$(a + b) + c =$ $a + (b + c)$
62	What other property under addition would be used to write $a + (b + c) = a + (c + b)$? ______	the commutative property

63 Recall that the empty set has no members or elements. If E is the empty set, then

$E = \{\ \}$.

If

$B = \{\text{James, Bill, Ellen, Harold}\}$

what is $B \cup E$?

$B \cup E =$ ______.

{James, Bill, Ellen, Harold}

64 In general, if E is the empty set and S is any set, then $S \cup E =$ ______. The empty set is called the identity element for the set operation union.

S

65 If $E = \{\ \}$, then $N(E) = 0$. Let $N(A) = a$. Since $A \cup E = A$, we may write the addition sentence ______.

$a + 0 = a$

66 By the commutative property under union and the property of the identity element E, $A \cup E = E \cup A = A$. Hence, we may write the addition sentence ______ = ______ = ______.

$a + 0 = 0 + a = a$

67 The number 0 is called the identity element for addition and means that if ______ is added to any number, the result will be that number.

0

68 If n is any whole number, then

$n + 0 =$ ______ = ______.

$0 + n = n$

Summary

In this sub-program we have developed three additional properties of the set of whole numbers under the operation of addition. In the previous sub-program we developed the closure property of the whole numbers under addition. This means that when we are adding whole numbers, these four properties will hold for any whole numbers. These properties are the following:

(1) For any whole numbers a and b, $a + b = c$, where c is a whole number (closure)

(2) For any whole numbers a and b, $a + b = b + a$. (commutativity)

(3) For any whole numbers a, b and c, $(a + b) + c = a + (b + c)$. (associativity)

(4) There is a unique whole number 0 such that $n + 0 = 0 + n = n$ for any whole number n. (identity element)

CHAPTER 6

SUBTRACTION AND ADDITION

We shall find it advantageous to present the operation of subtraction from three points of view, representing two fundamentally different approaches. The first approach is similar to the way we have defined addition, that is, in terms of sets and set operations. The second approach defines subtraction directly in terms of sets and set operations. The third approach defines subtraction directly in terms of addition of whole numbers; that is, as an inverse operation.

Approaching subtraction through sets and set operations is done in two different ways, one corresponding to a "take-away" operation, the other to an "add-to" operation.

6-1. Subsets and Remainder Sets

We have learned some things about sets. Now we need to develop two additional concepts: *subset* and *remainder set*. We first consider the notion of *subset*.

1	Consider A = {Mary, George, Bill, Ann, Tom, Allen} B = {Mary, Ann, Tom, Bill}. Every member of B ________ (is, is not) a member of A.	is
2	Given a pair of sets such as A and B. If every member of B also is a member of A, we say that B is a *subset* of ______.	A
3	Given the sets of Frame 1, we say that B is a ______ of A.	subset
4	Since every member of B is a member of B, we also can say that B is a ______ of B.	subset

5	Any set ______ (is, is not) a subset of itself.	is
6	Let E = {ball, bat, book}. Find a subset of E with the number property 3 and name this set F. F = ______.	{ball,bat,book}
7	Let G be a subset of {ball, bat, book} such that N(G) = 2. Then, G = ______, or G = ______, or G = ______.	{ball,bat} {ball,book} {bat,book}
8	We designate that C is a subset of D by the expression $C \subset D$ and means that every member of C ______ (is, is not) a member of D. The symbol $\subset$ is read "is a subset of."	is
9	Let C be the set of all children in a given class, and D be the set of all boys in the same class. Then D is a ______ of C.	subset
10	D ______ (is, is not) a subset of C if there is no element of C which is not an element of D.	is
11	Since there is no element of the empty set that is not an element of D, the empty set is a ______ of D.	subset
12	{ } is a subset of G = {______}.	For example, {ball,book}
13	{ } is a ______ of every set.	subset

14 The statement,

"The empty set is a subset of A,"

can be written symbolically as ______.

$\{\ \} \subset A$

15 The statement,

"Any set A is a subset of itself,"

can be written symbolically as ______.

$A \subset A$

16 Consider $R = \{0, 1, 2, 3\}$. Which of the following do not represent subsets of R? (Check all correct responses.)

☐ (a) $\{0, 1, 2, 3\}$

☐ (b) $\{\ \}$

☐ (c) $\{0, 5\}$

☐ (d) $\{1, 2\}$

☐ (e) $\{0, 3, 2\}$

16(a) Incorrect, since every set is a subset of itself.

16(b) Incorrect, since the empty set is a subset of every set.

16(c) Correct. $\{0, 5\}$ is not a subset of R since 5 is not a member of R.

16(d) Incorrect. Since every member of $\{1, 2\}$ is a member of R, $\{1, 2\}$ is a subset of R.

16(e) Incorrect. Since every member of $\{0, 1, 2\}$ is a member of R, $\{0, 3, 2\}$ is a subset of R.

17 Which of the following represent subsets of

$$A = \{a, b, c, d, e\}:$$

(a) {a, b, g}

(b) b, c, d

(c) {d, c}

17(a) Incorrect, since g is not a member of A, but is a member of {a, b, g}.

17(b) This is a listing and by agreement does not represent a set. Hence it cannot be a subset.

17(c) Correct. Each member of {d, c} is a member of A = {a, b, c, d, e}.

The notion of proper subset is not germane to the development of subtraction from sets, but on occasion is a useful concept and hence will be introduced. The reader can proceed to Frame 24 if he so chooses without any loss of continuity in the development of subtraction.

18	A proper subset of A is a ______ of A that has some, but not all of the members of A as its members.	subset
19	A proper subset ____________ (can, cannot) be empty.	cannot
20	The number property of a proper subset of A is less than the ______ property of A.	number

21 Consider T = {2, 0, 1}. Which of the following represents a proper subset of T?

☐ (a) {0, 1, 2}

☐ (b) {0, 2}

☐ (c) {1, 3}

21(a) Incorrect. {0, 1, 2} is a subset of T, but it contains all of the elements of T so it is not a proper subset of T.

21(b) Correct, since {0, 2} contains some but not all the elements of T.

21(c) Incorrect. {1, 3} is not a subset of T since it contains an element that is not in T.

22 Consider U = {2, 3, 1, 0}. Which of the following represent proper subsets of U?

☐ (a) { }

☐ (b) {0}

☐ (c) {0, 3}

22(a) Incorrect. Since the empty set contains no elements, it is not a proper subset of U.

22(b) Correct. {0} contains some, but not all of the elements of U and 0 is a member of U.

22(c) Correct. {0, 3} contains some, but not all of the members of U.

23 Which of the following subsets are not proper subsets of X = {4, 2, 0}?

☐ (a) { }

☐ (b) {4, 0, 2}

☐ (c) {0, 4}

23(a) Correct. A proper subset of X must contain some but not all elements of X. { } contains no elements of X.

23(b) {4, 0, 2} contains all the elements of X, contrary to the definition of proper subset. Hence, this response is correct.

23(c) Incorrect. {0, 4} contains some but not all of the elements of X and is a proper subset of X.

Let us now consider remainder sets, a concept which underlies one approach to subtraction.

24 Let A denote the set of letters used to spell the word "contract". Hence,

A = ______. {c,o,n,t,r,a}

25 The set of letters used to spell the word "attract" is

B = ______. {t,r,a,c}

26 Consider the sets of Frames 24-25 B is a subset of A and the set of elements of A which are not elements of B is ______. {o,n}

27 The set of elements of A which are not elements of B, where B is a subset of A, is called the remainder set of B with respect to A and is designated by $A \sim B$. Hence, for A and B, the sets of Frames 24-25,

$A \sim B =$ ______.

$\{o, n\}$

28 If N is a subset of M, the set consisting of the elements of M not elements of N is the ______ set of N with respect to M.

remainder

29 We denote the remainder set of N with respect to M by ______. The symbol $\sim$ is read "wiggle".

$M \sim N$

30 The remainder set of N with respect to M may be found by "taking away" the ______ of N from M.

members or elements

31 Let A = set of letters used to spell the word "contract"

B = set of letters used to spell the word "attract"

$F = \{a\}$.

Then, $\{t, r, c\}$ may be indicated by the following: (Check one.)

☐ (a) $A \sim F$ ☐ (b) $F \sim B$ ☐ (c) $B \sim F$

31(a) Incorrect, since $A \sim F = \{c, o, n, t, r\}$.

31(b) Incorrect. Since B is not a subset of F, $F \sim B$ has no meaning.

31(c) Correct. $B \sim F = \{t, r, c\}$ is the remainder set of F with respect to B.

32 If G is a subset of H, indicate symbolically the set of elements of H which are not elements of G. ______ — H ~ G

33 The set T = R ~ S is called the ______ set of S with respect to R. — remainder

6-2. First Definition of Subtraction

We now define subtraction in terms of sets, subsets, and remainder sets.

34 If A = {book, pen, dog, bottle, box}, then, N(A) = ______. — 5

35 Let B = {pen, bottle, box}. Then B is a ______ of A and N(B) = 3. — subset

36 If C = A ~ B, then C = ______. — {book,dog}

37 N(A ~ B) = N(C) = ______. — 2

38 We then say that N(A) - N(B) = 5 - 3 = N(A ~ B) = ______. — 2

39 Consider the numbers 5 and 2. Choose D = {a, b, c, d, e} and a subset of D such as E = {b, d}. Since N(D) = 5 and N(E) = 2, it follows that 5 - 2 = 3 because:

- ☐ (a) N(D ~ E) = N({a, c, e}) = 3
- ☐ (b) N(E ~ D) = N({a, c, e}) = 3
- ☐ (c) N(D ~ E) = N({a, b, e}) = 3

39(a) Correct.

39(b) Incorrect. Reread Frames 24-33.

39(c) Incorrect. Reread Frames 24-33.

The above illustrates our first definition of subtraction. Let c denote a number and d a number less than or equal to c ($d \leq c$). We obtain the remainder, or the result of subtracting d from c, denoted by c - d, as the number property of a remainder set. Arbitrarily choose a set C such that N(C) = c. Next choose a set D such that D is a subset of C and N(D) = d. Then c - d = N(C ~ D).

40 Given the numbers 3 and 1. Choose A = {x, t, y} and B = {t}. Since the remainder set A ~ B = {x, y}, 3 - 1 = N(A ~ B) = ______.

2

41 Given the numbers a = 3 and b = 2. If A = {r, t, s}, then B may be the set ______ or ______ or ______.

{r,s} or {s,r}
{t,s} or {s,t};
{t,r} or {r,t}

42 Using A, the set given in Frame 41, and your response for B in Frame 41, A ~ B =

☐ (a) {r} ☐ (b) {s} ☐ (c) {t}

42(a) Correct, if your set B = {s, t} or {t, s}.

42(b) Correct, if your set B = {r, t} or {t, r}.

42(c) Correct, if your set B = {r, s} or {s, r}.

43 If a = 8, b = 5 and the set B is chosen so that B = {q, r, s, u, t}, then for N(A ~ B) = 3, set A could be: (Check one.)

☐ (a) A = {m, n, o, p, r, s, u, t}
☐ (b) A = {s, u, t, p, q, r, m, n}
☐ (c) A = {q, r, s, t, u, m, n}
☐ (d) A = {q, r, s}

43(a) Incorrect. B is not a subset of A.

43(b) Correct. B is a subset of A and N(A) = 8.

43(c) Incorrect. B is a subset of A, but N(A) = 7.

43(d) Incorrect. {q, r, s} has the number property 3, not 8.

44	Given $R = \{\text{elm, oak, cedar}\}$ $S = \{\text{cedar, elm, oak}\}$. Then S ________ a subset of R. (is, is not)	is
45	$R \sim S =$ ______.	the empty set or { }
46	$N(R \sim S) =$ ______.	0
47	Hence $3 - 3 = N($______$)$ or ______.	$N(R \sim S)$; 0
48	Given $A = \{a, c, d\}$ and $B = \{\ \}$. Then, B ________ a subset of A. (is, is not)	is
49	Using sets A and B of Frame 48, $A \sim B =$ ______.	{a, c, d}
50	$N(A) = 3$, $N(B) =$ ______ and $N(A \sim B) =$ ______.	0; 3
51	It follows that $3 - 0 =$ ______.	3

52 If $R = \{x, t, v, y\}$ and $S = \{\ \}$, then $R \sim S =$ (Check one.)

☐ (a) R ☐ (b) S ☐ (c) $R \sim S$

52(a) This is a correct response and is the most economical way to express the set $R \sim S$.

52(b) This is not a correct response. Return to the definition of ~ in Frame 24 and continue therefrom.

52(c) This is a correct response since = means the same set is named. However, 52(a) is more desirable.

53 We observe from Frame 49 that A ~ { } = A
and from Frame 52 that R ~ { } = ______. R

54 If S is any set whatsoever, then
S ~ { } = ______. S

55 It follows that for any set S,
N(S ~ { }) = N(______). N(S)

56 If N(A ~ B) = N(A) for any set A. Then

☐ (a) B = 0 ☐ (b) B = {0} ☐ (c) B = { }

56(a) Incorrect. N(B) = 0, but B = { }.

56(b) Incorrect. If an element is placed within the braces, we do not have the empty set.

56(c) Correct.

57 Let N(A) = 5. Since N(A ~ { }) = N(A),
it follows that 5 - 0 = ______. 5

58 Similarly, 3 - 0 = ______. 3

59 If n is any number, then n - 0 = ______. n

The foregoing definition of subtraction is given in terms of a set, a subset of this set, and the remainder set. This definition of subtraction justifies the "take-away" method of subtracting b from a if $b \leq a$. Two special cases considered were (1) if a = b, then a - b = b - a = 0 and (2) if b = 0, then a - b = a - 0 = a.

6-3. Second Definition of Subtraction

In order to introduce a second definition of subtraction we will use the union of disjoint sets in a manner similar to that used in addition.

60	Let $K = \{\text{boat, kite, ball, bat, doll}\}$ $J = \{\text{boat, ball, bat}\}$. If P denotes the remainder set, then $P = K \sim J =$ ______.	{kite,doll}
61	J and P are ______ since they have no elements in common.	disjoint
62	We also note that $J \cup P =$ ______.	{boat,kite,ball, bat,doll}
63	Similarly $P \cup J =$ ______.	{boat,kite,ball, bat,doll}
64	If $J \cup P = K$, it follows that $J \cup (K \sim J) =$ ______.	K
65	Since union is commutative $J \cup (K \sim J) = (K \sim J) \cup$ ______.	J

One fact suggested by Frame 64 and Frame 65 is that the operations of union of disjoint sets and wiggle are *inverse* operations. Another pair of inverse operations is addition and subtraction of numbers.

Instead of using remainder sets we may define subtraction as follows:

Definition: Let $N(A) = a$, $N(B) = b$, and $N(C) = c$. Then, $a - b = c$ if and only if $N(A) = N(B \cup C)$ with B and C disjoint.

66 Let $P = \{a, b, c, d,\}$, $Q = \{b, d, e\}$, $p = N(P)$ and $q = N(Q)$. Then $p - q = N(R)$ where: (Check one.)

☐ (a) $R = \{x\}$

☐ (b) $R = \{a, e\}$

☐ (c) $R = \{a, c\}$

66(a) This response is correct since Q and R are disjoint and $N(Q \cup R) = 4 = N(P)$.

66(b) This response is incorrect even though $N(Q \cup R) = 4$, since Q and R are not disjoint.

66(c) This response is incorrect even though Q and R are disjoint, since $N(Q \cup R) = 5$, not 4.

67 In this and the following frames consider

A = { ★ , + , ⬬ , ◇ , ▲ }

B = { ○ , ◇ , ⬬ }

C = { ▲ , ★ }

D = { ◇ , ★ }

B ∪ C = {______}.

{ ○ , ◇ , ⬬ , ▲ , ★ }

68 N(B ∪ C) = ______.

5

69 B ∪ D = {______}.

{ ○ , ◇ , ⬬ , ★ }

70 N(B ∪ D) = ______.

4

71 Using the sets of Frame 67, $N(A) =$ (Check one.)

(a) $N(B \cup C)$ (b) $N(B \cup D)$ (c) $N(C \cup D)$

71(a) Correct, since B and D are disjoint and $N(B \cup C) = 5$.

71(b) Incorrect. $N(B \cup D) = 4$, since B and D are not disjoint. Note that $N(B) + N(D)$ does equal 5, however.

71(c) Incorrect. $N(C \cup D) = 3$ and $N(A) = 5$.

72	Since A matches $B \cup C$, $N(A) = N(\underline{\quad\quad})$.	$N(B \cup C)$
73	$N(A) - N(\underline{\quad\quad}) = N(C)$.	$N(B)$
74	Since A matches $C \cup B$, $N(A) = N(\underline{\quad\quad})$.	$N(C \cup B)$
75	If $N(A) = 5$, $N(B) = 3$, $N(C) = 2$, and B and C are disjoint as in Frame 67, then $5 - 3 =$ ____ and	2
	$5 - 2 =$ ____.	3

In this definition of subtraction we select sets A and B such that $N(A) = a$ and $N(B) = b$ $(b \leq a)$. We choose another set C such that B and C are disjoint and $B \cup C$ matches A. Then $N(B \cup C) = N(A)$. This is equivalent to finding a number c which if added to b gives a.

76	{ } has no elements in common with the set B. Hence, { } and B are ______.	disjoint
77	$B \cup \{\ \} =$ ______.	B
78	$B = \{\ \} \cup$ ______.	B

79	$N(B) = N(\{\ \} \cup \_\_\_\_\_)$.	B
80	It now follows that $N(B) - N(\{\ \}) = \underline{N(\qquad)}$.	N(B)
81	A second conclusion is that $N(B) - N(B) = \underline{N(\qquad)}$.	N({ })
82	Again we arrive at the conclusion that if n is any number, then $n - 0 = \_\_\_\_$ and $n - n = \_\_\_\_$.	n; 0

6-4. Third Definition of Subtraction

The third definition of subtraction is closely related to the definition just developed. In the second definition we sought to find a disjoint set with the appropriate number property. Instead of working with sets we now define subtraction in terms of addition as follows:

Definition: $a - b = c$ if and only if $a = b + c$.

83	$5 - 1 = 4$ since $1 + 4 = \_\_\_\_$.	5
84	Since $5 - 3 = 2$, $5 = \_\_\_\_ + 2$.	3
85	$8 - \_\_\_\_ = 3$, since $8 = 5 + 3$.	5
86	Since $9 = 6 + 3$, $9 - 6 = \_\_\_\_$.	3
87	Since $15 = 7 + 8$, then $15 - \_\_\_\_ = 8$.	7
88	$12 = 5 + 7 = 7 + 5$. Hence $12 - 5 = \_\_\_\_$ and $12 - 7 = \_\_\_\_$.	7 5

89 In working with whole numbers 3 - 5 = (Check one.)

☐ (a) 2 ☐ (b) 0 ☐ (c) not possible

89(a) Incorrect, since 5 + 2 = 7, not 3.

89(b) Incorrect, since 5 + 0 = 5, not 3.

89(c) Correct. There is no whole number which if added to 5 gives 3.

CHAPTER 7

ADDITION AND SUBTRACTION TECHNIQUES

In this chapter we see the manner in which the properties of whole numbers are used in computational techniques of addition and subtraction. There are no new concepts introduced.

1	6 + 3 = ______ .	9
2	Then, 36 + 3 may be written as (3 × 10) + ______ + 3 .	6
3	Or, using associativity, as (3 × 10) + 6 + ______ .	3
4	Or as (3 × 10) + ______ = 39 .	9
5	Write 236 in expanded notation: ______	(2 × 100) + (3 × 10) + 6
6	The sum 236 + 3 may be written as (2 × 100) + (3 × 10) + (____ + 3) .	6
7	Then, (2 × 100) + (3 × 10) + ______ = 239 .	9
8	9 + 5 = ______	14
9	Furthermore, 14 = (1 × 10) + ______ .	4
10	Then, 69 + 5 may be written as (6 × 10) + (____ + 5) .	9
11	69 + 5 = (6 × 10) + ______	14
12	(6 × 10) + 14 = (6 × 10) + (1 × 10) + ______	4
13	(6 × 10) + (1 × 10) + 4 may be written as [(6 + 1) × ____] + 4 .	10
14	[(6 + 1) × 10] + 4 = (____ × 10) + 4	7
15	Finally, 70 + 4 = ______ .	74

16	To find the sum of 44 and 25 we could write [(4 × 10) + ____] + [(____) + 5] .	4; (2 × 10)
17	Then, by using commutativity and associativity, this becomes [(4 × 10) + (____)] + (4 + 5) .	(2 × 10)
18	[(4 × 10) + (2 × 10)] + (4 + 5) may be written as [(4 + 2) × 10] + ____ .	9
19	Hence, 44 + 25 = (____ × 10) + 9 = ______	6 69
20	To add 38 and 46, write [(3 × 10) + 8] + [(____) + ____] .	(4 × 10) + 6
21	Then, rewrite this as [(3 × 10) + (4 × 10)] + (____ + ____) = [(3 × 10) + (4 × 10)] + ______ .	8 + 6 14
22	This may be written as [(3 + 4) × 10] + [(____ × 10) + 4] .	1
23	Or as [(3 + 4 + 1) × 10] + ____ .	4
24	Thus, 38 + 46 = ____ × 10 + 4 = ______	8 84
25	Find the sum of 276 and 398 by writing 276 as (2 × ____) + (7 × ____) + 6 and 398 as (____ × 100) + (____ × 10) + ____	100; 10 3; 9; 8
26	Then, 276 + 398 = (2 × 100) + (7 × ____) + (3 × 100) + (____ × 10) + (6 + 8)	10 9
27	Using the properties of whole numbers, the above becomes [(2 + 3) × ____] + [(7 + 9) × ____] + 14 .	100; 10

28	It now follows that 276 + 398 = (5 × 100) + (16 × 10) + (____ × 10) + 4 .	1
29	This becomes (5 × 100) + [(16 + 1) × 10] + 4 = (5 × 100) + (____ × 10) + 4 .	17
30	This now is rewritten as (5 × 100) + [(10 + 7) × 10] + 4 = (5 × 100) + [(1 × ____) + (7 × 10)] + 4 .	100
31	Hence, 276 + 398 = [(____ + ____) × 100] + (7 × 10) + 4 = 674 .	(5 + 1)
32	The sum of 276 and 398 could be written as $\begin{array}{r}276\\ \underline{+\ 398}\end{array}$ And in adding, the sum of 8 and 6 is ______ or 1 ______ and 4 .	14 ten or 10
33	Add the 1 ten to (7 + 9) tens which gives 17 tens or (10 + ____) tens.	7
34	10 tens is the same as ____ hundred.	1
35	Add the 1 hundred to (2 + 3) hundreds which gives 6 hundreds. Hence, $\begin{array}{r}276\\ \underline{+\ 398}\end{array}$ will give the sum ______.	674
36	To find the difference 57 - 22 write 57 - 22 as (50 + 7) - (20 + ____) = (50 - 20) + (7 - ____) = 30 + 5 = 35	2 2

37 Writing 57 - 22 in vertical form as

$$\begin{array}{rr} & 50 + 7 \\ \text{subtract} & 20 + 2 \\ \hline \end{array}$$

it follows that 57 - 22 = ____ + ____ 30 + 5

= ______ 35

38 Consider the difference 52 - 27 .

Write 52 as ____ + 2 50

and 27 as 20 + 7 .

39 In order to subtract 20 + 7 from 50 + 2 ,
50 + 2 may be written as 40 + ____ + 2 . 10

40 Now 40 + 10 + 2

subtract 20 + 7

$$\begin{array}{rl} \text{Now} & 40 + 10 + 2 \\ \text{subtract} & 20 + 7 \\ \hline \text{gives} & \_\_ + \_\_ + 2 = 25 \end{array}$$

gives ___ + ___ + 2 = 25 . 20 + 3

CHAPTER 8

PROPERTIES OF WHOLE NUMBERS UNDER THE OPERATION MULTIPLICATION

8-1. Addition (Review)

We have learned that addition is a binary operation on whole numbers. For any two whole numbers the operation of addition, defined in terms of the union of disjoint sets, results in another whole number.

1	If B and C denote sets, then the ______ of B and C has as its elements those elements which belong to B or to C or to both B and C.	union
2	Since disjoint sets contain no elements in common, their union contains all of the ______ in each of the sets.	elements
3	The number property of $B = \{\text{Jane, George, Bob, Bill}\}$ is ______.	4 or four
4	A numeral for the number property of $C = \{\text{Ellen, Ann, Albert}\}$ is ______.	3 or three
5	For the sets of Frames 3 and 4, $B \cup C =$ ______.	{Jane, Ellen, Ann, Albert, Bob, Bill, George}
6	The number property of $B \cup C$ is ______.	seven or 7
7	$N(B) =$ ______. $N(C) =$ ______. $N(B \cup C) =$ ______.	four or 4 three or 3 seven or 7

8	Since B and C have no elements in common, then B and C are ______ sets.	disjoint

8-2. The Operation Multiplication

Multiplication may be defined in terms of the union of disjoint matching sets.

9	Given three disjoint sets of trees to be planted on the school ground: A = {elm, oak, birch, hickory} B = {plum, apple, pear, peach} C = {hackberry, maple, chestnut, willow} The total number of trees may be found by counting. The number of trees is ______.	12
10	By one-to-one correspondence between the elements, we find that A ______ B.	matches
11	Likewise, B matches C, and by the transitive property A matches ______.	C
12	Since A, B and C are matching sets, then they all have the same ______ property.	number
13	$N(A) = N(B) = N(C) =$ ______.	4 or four
14	Addition and the associative property may be used to find the number of trees in Frame 12. Hence, $(4 + 4) + 4 = 8 + 4 =$ ______.	12
15	The total number of trees in A, B and C is the ______ ______ of $(A \cup B) \cup C$.	number property
16	$N[(A \cup B) \cup C] = N($______$) + N(C) = 12.$	$A \cup B$

17	The trees may be arranged in a rectangular array of 3 rows with 4 trees in each row as follows: x x x x x x x x x x x x The above array illustrates the number sentence 3 × 4 = ______.	12
18	Arrange the trees in a rectangular array of 4 rows with 3 trees in each row. The above array illustrates the number sentence 3 + 3 + 3 + 3 = 4 × ______ = 12.	x x x x x x x x x x x x 3
19	Given the numbers 2 and 5. Exhibit a 2 by 5 rectangular array of objects such that there are 2 rows and 5 columns. The array illustrates the number sentence ____ × ____ = 10 .	x x x x x x x x x x 2 × 5 = 10

20	The ______ of objects in the rectangular array of Frame 19 is represented by (2 × 5). The number of (2 × 5) is called the product of the factors 2 and 5.	number
21	Arrange ten objects in a rectangular array of 5 rows with 2 objects in each row. The array illustrates the number sentence ____ × ____ = 10 .	x x x x x x x x x x 5 × 2 = 10
22	Since the number of objects in the rectangular array of Frame 19 is the same as the number of objects in the rectangular array of Frame 21, it follows that ____ × ____ = ____ × ____.	2 × 5 = 5 × 2
23	Let m and n denote whole numbers. The total number of objects in a rectangular array of m rows and n columns is denoted by (____ × ____). The number (m × n) is called the product of the factors m and n.	(m × n)
24	The total number of objects in a rectangular array of n rows and m columns is denoted by the product (____ × ____) of the factors n and m.	(n × m)
25	Since the number property for the total set of objects in the rectangular arrays for Frames 23 and 24 is the same, m × n = n × m and the whole numbers are ______ under the operation of multiplication.	commutative

26	If p and q are any whole numbers, then the product $(p \times q)$ can be expressed as a p by q rectangular array of objects. The number property of the p by q array of objects is itself a ______ number.	whole
27	Since the product of two whole numbers is a whole number, that is, a member of the set of whole numbers, the set of whole numbers is ______ under the operation of multiplication.	closed

In this sub-program the idea that multiplication of two whole numbers is basically the union of disjoint sets, each with the same number property, has been considered. The product (3×5) of the factors 3 and 5 may be associated with the sum $[(5 + 5) + 5]$ and also may be thought of as the number property of a rectangular array of three rows with five objects in each row. The 3 by 5 array is the set of objects in

x x x x x
x x x x x
x x x x x

and illustrates the number sentence $3 \times 5 = 15$.

The preceding is a method of associating a single whole number with the product $(m \times n)$ of two whole numbers m and n. Hence, the operation of multiplication is defined for any two whole numbers. The set of whole numbers is closed under the operation of multiplication. Since $m \times n = n \times m$ for any whole numbers m and n, the operation of multiplication is commutative.

8-3. The Associative Property under Multiplication

Recall that the set of whole numbers has the associative property with respect to addition. Now the associative property of the whole numbers with respect to multiplication will be considered.

28	Consider the product $3 \times 4 \times 2$. The expression $(3 \times 4) \times 2$ means that the product (______) is to be considered first, then this product is to be multiplied by 2.	(3×4)
29	Since the operation of multiplication is ______ for whole numbers, the product of any two whole numbers is a whole number.	closed
30	In Frame 28, if (3×4) is replaced by 12, then the product $(3 \times 4) \times 2$ becomes ___ × ___ = 24.	12×2
31	If the product $3 \times 4 \times 2$ is considered as $3 \times (4 \times 2)$, one obtains ___ × ___ = 24.	3×8
32	For the product $3 \times 4 \times 2$ we may write $(3 \times 4) \times 2 =$ 3 × (___ × ___) = ______ .	$3 \times (4 \times 2) = 24$
33	Consider the product $2 \times 5 \times 3$. $(2 \times 5) \times 3$ ______ (is, is not) equal to $2 \times (5 \times 3)$.	is
34	The product in Frame 33 and its solution is an illustration of the ______ property under multiplication.	associative
35	Given the three whole numbers 8, 4, 5. Then, $8 \times 4 \times 5 = (8 \times 4) \times$ ______ = ___ × ___ = ______ . And, $8 \times 4 \times 5 =$ ___ × (___ × ___) = ___ × ___ = ______ . Hence, $(8 \times 4) \times 5 = 8 \times (4 \times 5)$.	 $(8 \times 4) \times 5$ 32×5 160 $8 \times (4 \times 5)$ 8×20 160

36 Let a, b, c denote any triple of whole numbers. Write the associative property under multiplication. ______

$(a \times b) \times c = a \times (b \times c)$
(This statement is true for all whole numbers a, b, c.)

The preceding sub-program presented the associative property of the set of whole numbers under the operation of multiplication. This property corresponds to the associative property of addition developed earlier. A formal statement of the associative property is as follows:

> If a, b and c are any three whole numbers,
> then $(a \times b) \times c = a \times (b \times c)$.

8-4. Rearrangement Using Commutativity and Associativity

The whole numbers are both commutative and associative with respect to multiplication. Now consider using the commutative and associative properties in combination with each other in the rearrangement of the factors of a product.

37 Consider the product $7 \times 4 \times 5$. If 7 and 4 are associated together as (7×4), then by the commutative property under multiplication we may write

$$7 \times 4 \times 5 = (7 \times 4) \times 5$$
$$= \underline{(\quad \times \quad) \times 5}$$
$$= \underline{\quad \times \quad \times \quad}.$$

$(4 \times 7) \times 5$

$4 \times 7 \times 5$

38 The use of the commutative property in a series of factors enables us to rearrange the factors. Another rearrangement of the three factors in Frame 37 is

$$7 \times 4 \times 5 = 7 \times (4 \times 5)$$
$$= \underline{\quad \times (\quad \times \quad)}$$
$$= \underline{\quad \times \quad \times \quad}$$

$7 \times (5 \times 4)$

$7 \times 5 \times 4$

39 Make two distinct rearrangements on the order of the factors in the product $2 \times 4 \times 3 \times 5$ by use of the commutative property.

$2 \times 4 \times 3 \times 5 =$ ______

$2 \times 4 \times 3 \times 5 =$ ______

$2 \times 3 \times 4 \times 5$

$4 \times 3 \times 2 \times 5$

By the use of the commutative and the associative properties of multiplication, it is possible to rearrange a set of factors in many different ways.

8-5. The Roles of 1 and 0 in Multiplication

The number 1 plays, with respect to multiplication, a role analogous to that played by 0 with respect to addition.

40 x x x x

This array represents the number sentence ______.

$1 \times 4 = 4$

41 x
x
x
x

This array represents the number sentence ______.

$4 \times 1 = 4$

42 Since the number of elements in each of the above arrays is the same, namely four, the number sentence is written as

$1 \times 4 =$ ___ × ___ .

4×1

43 The number of objects in a 1 by n array ______ (is, is not) the same as the number of objects in an n by 1 array. Hence,
$1 \times n = n \times 1 = n$ for any whole number n.

is

In the set of whole numbers the number 0, besides playing the role of the identity element for addition, also has a rather special property with respect to multiplication.

44	The set of elements in a 0 by 5 array ______ (is, is not) empty.	is
45	Hence, $0 \times 5 =$ ______.	0
46	The set of elements in a 5 by 0 array is ______.	empty
47	Hence, $5 \times 0 =$ ______.	0
48	0×5 ______ (=, ≠) 5×0.	=
49	$0 \times n = 0$ for any whole number ______.	n
50	$n \times 0 =$ ______ for any whole number n.	0
51	Thus, for any whole number n, $n \times 0 = 0 \times n =$ ______.	0

52 The identity element for multiplication in the set of whole numbers is the number

☐ (a) 0 ☐ (b) 1 ☐ (c) n

52(a) Incorrect. 0 is the identity element for addition in the set of whole numbers, but is not the identity element for multiplication.

52(b) Correct, since $1 \times n = n \times 1 = n$ for any whole number n.

52(c) Incorrect. See 52(b).

8-6. The Distributive Property

We have seen that multiplication may be described as repeated addition. For example, $3 \times 7 = 7 + 7 + 7 = 21$. Another important property that links the two operations addition and multiplication is the distributive property of multiplication over addition.

53	x x x x x x x x x x x x This array represents the product (___ × ___).	(3 × 4)
54	This array represents the product (___ × ___). x x x x x x x x x x x x x x x	(3 × 5)
55	The following array represents the product ____ × (4 + 5). x x x x | x x x x x x x x x | x x x x x x x x x | x x x x x	3
56	(3 × 4) + (3 × 5) = 12 + ____ = ____.	15; 27
57	3 × (4 + 5) = 3 × ____ = ____.	9; 27
58	(3 × 4) + (3 × 5) $\underset{(=,\ \neq)}{\underline{\qquad}}$ 3 × (4 + 5).	=
59	(7 × 5) + (7 × 2) = ____ + ____ = 49.	35 + 14
60	7 × (5 + 2) = ____ × ____ = 49.	7 × 7
61	7 × (5 + 2) $\underset{(=,\ \neq)}{\underline{\qquad}}$ (7 × 5) + (7 × 2).	=
62	Frames 53 - 61 suggest that a × (b + c) = (a × b) + (___ × ___) for any three whole numbers a, b, c. The sentence a × (b + c) = (a × b) + (a × c) is called the distributive property of multiplication over addition.	(a × c)

Summary

In this chapter we have considered the binary operation of multiplication and its properties. These properties may be summarized as follows:

(1) For any whole numbers a and b, $a \times b = n$, where n is a whole number. (closure)

(2) For any whole numbers a and b, $a \times b = b \times a$. (commutativity)

(3) For any whole numbers a, b and c, $(a \times b) \times c = a \times (b \times c)$. (associativity)

(4) There is a unique whole number 1 such that $n \times 1 = 1 \times n = n$ for any whole number n. (identity element)

(5) For any whole number n, $n \times 0 = 0 \times n = 0$. (multiplication property of 0)

(6) For any whole numbers a, b and c, $a \times (b + c) = (a \times b) + (a \times c)$. (distributivity)

CHAPTER 9

DIVISION

9-1. Division of Whole Numbers

In the preceding chapter a rectangular array of a rows with b members in each row was used as a physical model for the product (a × b). From this and other models, the properties of multiplication for whole numbers were developed. The whole numbers under multiplication have the properties of closure, commutativity and associativity, and multiplication is distributive over addition. Also, the numbers 1 and 0 were found to have special properties, that is

$$1 \times a = a \qquad 0 \times a = 0$$
$$a \times 1 = a \qquad a \times 0 = 0$$

Division, the subject of this chapter, is related to multiplication in much the same way that subtraction is related to addition. First, we review multiplication as modeled by arrays and then, using the same model, develop division.

1 The array

$$\begin{array}{c|ccc} & 3 & & \\ \hline 4 & x & x & x \\ & x & x & x \\ & x & x & x \\ & x & x & x \end{array}$$

illustrates the number sentence $4 \times 3 =$ ______. 12

2 The array

$$\begin{array}{c|cccc} & 4 & & & \\ \hline 3 & x & x & x & x \\ & x & x & x & x \\ & x & x & x & x \end{array}$$

illustrates the number sentence ______ $\times 4 = 12$. 3

3 The array

$$\begin{array}{c|ccccc} & 5 & & & & \\ \hline 2 & x & x & x & x & x \\ & x & x & x & x & x \end{array}$$

illustrates the number sentence $2 \times$ ______ $= 10$. 5

4 Draw an array which illustrates the number sentence $3 \times 2 = 6$.

	2
3	x x x x x x

5 Draw an array which illustrates the number sentence $2 \times 3 = 6$.

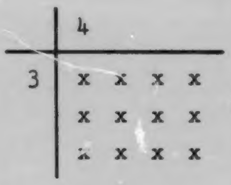
	3
2	x x x x x x

It should be observed that a notational convention has been adopted in writing these arrays. If the arrays do not agree exactly with the form of the array as given in the responses to Frames 4 - 5, the reader should review the chapter on multiplication (Chapter 8). The manner of writing these arrays plays an important role in the development of division.

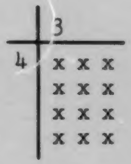
Division may be described as finding the unknown factor in a multiplication problem when the product and one factor are known.

6	If a and b are known whole numbers, then $a \div b = n$, read "a divided by b equals n," is a ______ sentence which says the same thing as $a = b \times n$.	number
7	Accordingly, if $b \neq 0$, division of a by b may be defined as follows: $a \div b = n$ if and only if ______. The number n is called the quotient.	$a = b \times n$
8	$12 \div 3 =$ ______ since $12 = 3 \times 4$.	4
9	Since $12 = 4 \times 3$, $12 \div 4 =$ ______.	3

10 In terms of an array, the number sentence
12 ÷ 3 = ______ is represented by — 4


4
3
x x x x
x x x x
x x x x

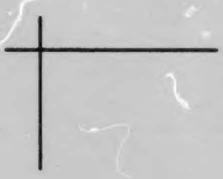
while the number sentence 12 ÷ 4 = 3 is represented by

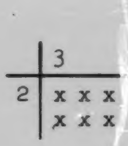

3
4
x x x
x x x
x x x
x x x

11 Since 20 = 5 × 4, 20 ÷ ______ = 4. — 5

12 Since 20 = 4 × 5, 20 ÷ ______ = 5. — 4

13 Arrange the elements of {x,x,x,x,x,x} into
an array which illustrates 6 ÷ 2 = 3.





Your attention is called to the order indicated in these arrays as used for division. The order is of utmost importance and must be preserved. Be sure that your answers agree exactly with those given in the program before continuing.

14 Arrange the elements of {x,x,x,x,x,x,x,x} into an array which illustrates 8 ÷ 2 = 4.

$$\begin{array}{c|cccc} & 4 & & & \\ \hline 2 & x & x & x & x \\ & x & x & x & x \end{array}$$

15 Arrange the elements of {x,x,x,x,x,x,x,x} into an array which illustrates 8 ÷ 4 = 2.

$$\begin{array}{c|cc} & 2 & \\ \hline 4 & x & x \\ & x & x \\ & x & x \\ & x & x \end{array}$$

16 Consider S = {x,x,x,x,x,x,x}. Which of the following is an array to illustrate N(S) ÷ 3?

☐ (a) $$\begin{array}{c|ccc} & & & \\ \hline & x & x & x \\ & x & x & x \\ & x & x & x \end{array}$$

☐ (c) $$\begin{array}{c|ccc} & & & \\ \hline & x & x & x \\ & x & x & \\ & x & x & \end{array}$$

☐ (b) $$\begin{array}{c|cc} & & \\ \hline & x & x \\ & x & x \\ & x & x \end{array}$$

☐ (d) none of these

16(a) Incorrect. This is an array, but it illustrates 9 ÷ 3, not 7 ÷ 3. N(S) = 7.

16(b) Incorrect. This is an array, but it illustrates 6 ÷ 3, not 7 ÷ 3. N(S) = 7.

16(c) Incorrect. This is not an array. (See Chapter 8.)

16(d) Correct.

17 In Frame 16 we considered several possibilities for trying to represent 7 ÷ 3, none of which worked. Which of the following explains why 7 ÷ 3 cannot be illustrated by a rectangular array?

☐ (a) I don't know.

☐ (b) $3 \times n \neq 7$ for any whole number n.

☐ (c) The set of whole numbers is not closed under the operation of division.

17(a) This possibly is correct, but we want a better answer than this.

17(b) This response is correct. 17(c) also is correct.

17(c) This response is correct. 17(b) also is correct.

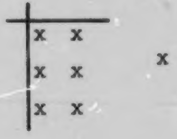
18 Instead of looking for a number n such that $7 = 3 \times n$, we consider the number sentence $7 = (3 \times 2) +$ ______.

1

19 In the number sentence $7 = (3 \times 2) + 1$, the number 2 is called the q______, and the number 1 is called the remainder.

quotient

20 We now illustrate the division 7 ÷ 3 by the arrangement



This is a rectangular array of three rows with two elements in each row indicating a ______ of 2 and a remainder of 1.

quotient

The following five frames consist of various arrangements. In each frame, complete the number sentence or write the corresponding number sentence, as appropriate.

21

x x x   x
x x x
x x x

10 = (3 × 3) + ______.

1

22

x x x x x
x x x x x   x x

12 = (2 ×   ) +   .

12 = (2 × 5) + 2

23

x x x x x
x x x x x   x
x x x x x   x
x x x x x   x

____ = (×   ) + 3.

23 = (4 × 5) + 3

24

x x x x
x x x x   x   x
x x x x   x
x x x x
x x x x

____ = (×   ) + ____.

23 = (5 × 4) + 3

25

x x x x x
x x x x x

____ = (×   ) + ____.

10 = (2 × 5) + 0

Arrangements help us to visualize the number sentences. In the next few frames, try to complete the sentences without drawing the arrangements.

26	17 = (5 × 3) + ____.	2
27	29 = (8 × ____) + 5.	3
28	13 = (____ × 2) + 1.	6
29	39 = (13 × ____) + 0.	3
30	39 = (3 × ____) + 0.	13
31	5 = (8 × ____) + 5.	0
32	4 = (5 × ____) + ____.	4 = (5 × 0) + 4

If we consider the two number sentences 19 = (6 × 3) + 1 and 19 = (6 × 1) + 13, both of which are true, we find that we prefer the first one and say that it is in the "best form" according to common practice.

33 In a problem of division such as 19 ÷ 6, we say that the sentence is in best form if the remainder is

☐ (a) a whole number less than the divisor 6.

☐ (b) any whole number.

33(a) This is correct, since in considering an arrangement, the remainder cannot be distributed into another column in the rectangular array of the arrangement.

33(b) This is incorrect, as it gives no basis for determining "best form" at all.

34 Which of the following number sentences is in the best form?

☐ (a) $19 = (6 \times 3) + 1$

☐ (b) $19 = (6 \times 2) + 7$

☐ (c) $19 = (6 \times 4) - 5$

34(a) This is in the best form since in the number sentence $n = (d \times q) + r$, the remainder r is less than the divisor d. Thus, $19 = (6 \times 3) + 1$ is in best form since $1 < 6$.

34(b) The sentence is true, but the remainder is not less than 6, hence it is not in the best form.

34(c) The sentence is true, but this sentence cannot be represented by an arrangement, an implied condition for best form.

35	The number sentence $15 = (5 \times 3) + 0$ may be written more simply as $15 = (\quad \times \quad)$.	(5×3)
36	The number sentence $9 = (9 \times 1) + 0$ may be written as $9 = (\quad \times \quad)$.	(9×1)
37	Since 0 is the additive identity element in the set of whole numbers, the number sentence $n = (d \times q) + 0$ may be written as $n = (\quad \times \quad)$.	$n = (d \times q)$
38	From the above frames we conclude that if the remainder is 0, it ________ (must, need not) be written in the number sentence.	need not

Zero is a very special number, and one must be very careful in using it. We have just observed that if the remainder is zero we need not write it. In the following frames, we point out other relationships involving zero.

39	If $n \neq 0$, then $n \div n =$ ____ since $n = n \times 1$.	1
40	This illustrates the mathematical fact that any nonzero number divided by itself yields a ______ of 1 (the identity of multiplication).	quotient
41	Therefore, $5 \div 5 =$ ____.	1
42	Since $n = 1 \times n$, $n \div 1 =$ ____.	n
43	In particular $7 \div 1 =$ ____.	7
44	In words the preceding two frames state that any whole number n divided by 1 gives as a quotient the number ____.	n
45	If $n \neq 0$, then $0 \div n =$ ____ since $0 = n \times 0$.	0
46	Frame 45 illustrates a fact with which many students have trouble, that is, if zero is divided by any ______ number, the quotient is zero.	nonzero
47	For example, $0 \div 5 =$ ____.	0

48 Which of the following number sentences are true?

(a) $0 = 0 \times 0$

(b) $0 = 0 \times 1$

(c) $0 = 0 \times n$, where n is any whole number.

Each is true. These number sentences illustrate the fact that $0 \div 0$ is an ambiguous symbol. If there were a unique number c such that $0 \div 0 = c$, then $0 = 0 \times c$ would be true only for this particular number c. But 48(c) shows us that $0 = 0 \times n$ for any whole number n, and not for just one particular one.

49 Which of the following number sentences are true?

☐ (a) $7 \div 0 = 0$

☐ (b) $7 \div 0 = 1$

☐ (c) $7 \div 0 = 7$

☐ (d) $7 \div 0$ is undefined

☐ (e) none of the above

49(a) This response is incorrect since $7 \neq 0 \times 0$.

49(b) This response is incorrect since $7 \neq 0 \times 1$.

49(c) This response is incorrect since $7 \neq 0 \times 7$.

49(d) This response is correct since the definition of division states if $b \neq 0$, $a \div b = c$ if and only if $a = b \times c$.

49(e) Incorrect. See 49(d).

In the two preceding frames we observed that $0 \div 0$ is an ambiguous symbol and that if $a \neq 0$, then $a \div 0$ does not represent any whole number at all. We use these observations as a reinforcement to our assertion that division by zero is not defined.

The number sentence $30 \div 6 = n$ states that n is a whole number such that $(6 \times n)$ will be the same as 30. Our knowledge of multiplication tells us that $n = 5$. In some cases, however, such as $7 \div 3 = n$, one cannot find a whole number n such that $7 = 3 \times n$. But $7 \div 3$ can be accomodated in the set of whole numbers by representing 7 as $(2 \times 3) = 1$. Thus one can say, "7 divided by 3 gives a quotient 2 and a remainder 1."

9-2. Properties of Division

In studying the operations of addition and multiplication, a number of properties were observed. In this part of the program we consider several properties of division, some of which may prove useful in developing computational techniques.

50 Since the sum or product of two whole numbers is always a ______ ______, the set of whole numbers is closed under the operations of addition and multiplication.

whole number

51 The set of whole numbers is not closed under either division or ______.

subtraction

52 The fact that division does not have the closure property can be illustrated by the following:

☐ (a) $6 \div 3 = n$ where n belongs to the set of whole numbers.

☐ (b) $6 \div 4 = n$ where n belongs to the set of whole numbers.

52(a) This response is incorrect since $6 \div 3 = 2$ and 2 is a whole number.

52(b) This response is correct since there is no whole number n such that $6 \div 4 = n$.

Any binary operation, which could be designated by the symbol $*$, is said to be commutative if $(a * b)$ is the same as $(b * a)$. For our purposes $*$ could be addition, multiplication, and so forth.

53 Multiplication is commutative in the set of whole numbers since $a \times b =$ ____ × ____.

b × a

9

54 Addition is commutative in the set of whole numbers since a + b = ______. b + a

55 Since $1 \div 1 = 1 \div 1$ and $5 \div 5 = 5 \div 5$, we conclude

☐ (a) the operation of division is commutative.

☐ (b) the operation of division is not commutative.

☐ (c) not enough evidence is provided to decide whether or not the operation of division, in general, is commutative.

☐ (d) division of a number by itself is commutative.

55(a) This response is incorrect. Two numerical examples are not sufficient to conclude the generality.

55(b) While the response is correct, the information given does not lead to this conclusion.

55(c) This response is correct on the evidence given.

55(d) This response is correct on the basis of the information given, but you should have checked 55(c) also.

56 Since $4 \div 2 \neq 2 \div 4$

☐ (a) The operation of division is commutative.

☐ (b) The operation of division is not commutative.

☐ (c) Not enough evidence is given to decide whether the operation of division is or is not commutative.

56(a) This response is incorrect. A basic property of division must be true for all cases and it is not true for this case.

56(b) This response is correct. A basic property must be true for all cases. Since $4 \div 2 \neq 2 \div 4$, we have one case for which it does not hold. Hence division is not commutative.

56(c) This response is incorrect. A basic property must be true for all cases and we have shown it not to hold for at least one case.

57	An operation $*$ is said to be associative if $a * (b * c) = (\_\_\_ * \_\_\_) * \_\_\_$.	$(a * b) * c$
58	Since $12 \div (6 \div 2) = 4$ and $(12 \div 6) \div 2 = 1$, it follows that division $\underset{\text{(is, is not)}}{\_\_\_\_\_\_}$ associative.	is not
59	We observed subtraction to be an inverse operation to addition. In a similar manner we observe division to be an inverse operation to ______.	multiplication
60	That is, $(3 \times 5) \div 5 = \_\_\_\_$.	3
61	$(0 \times 5) \div 5 = \_\_\_\_$.	0
62	$(9 \times 5) \div 5 = \_\_\_\_$.	9
63	In general, for any whole number b, $(b \times 5) \div 5 = \_\_\_\_$.	b
64	We also observe that $(n \times 2) \div 2 = \_\_\_\_$.	n
65	$(a \times 15) \div 15 = \_\_\_\_$.	a
66	In general, for any whole numbers a and b, $(a \times b) \div b = \_\_\_\_$, provided $b \neq 0$.	a
67	Thus, division by b can be thought of as the ______ of the operation of multiplication by b, provided $b \neq 0$.	inverse

A word of caution must be inserted here. In whole numbers, $(a \div b)$ may not even be defined. Hence $(a \div b) \times b$ also may not be defined. If $(a \div b)$ is defined, then $(a \div b) \times b$ is always a, provided $b \neq 0$.

68	$(15 + 24) \div 3 =$ ______ $\div$ ______ $=$ ______.	$39 \div 3 = 13$
69	$(15 \div 3) + (24 \div 3) =$ ______ $+$ ______ $=$ ______.	$5 + 8 = 13$
70	It follows that $(15 + 24) \div 3$ and $(15 \div 3) + (24 \div 3)$ are names for the same ______ number.	whole
71	This again generalizes: If $a \div n$ and $b \div n$ are whole numbers, then $(a \div n) + (b \div n) = ($ ______ $+$ ______ $) \div n$.	$a + b$
72	Since $(a + b) \div n = (a \div n) + (b \div n)$, division distributes over addition from the ______ (left, right) provided all quotients exist.	right
73	On the other hand, $6 \div (2 + 1) = 6 \div$ ______ $=$ ______.	$6 \div 3 = 2$
74	$(6 \div 2) + (6 \div 1) =$ ______ $+$ ______ $=$ ______.	$3 + 6 = 9$
75	$6 \div (2 + 1)$ and $(6 \div 2) + (6 \div 1)$ are names for ______ (the same, different) whole numbers.	different
76	It follows that division ______ (does, does not) distribute over addition from the left.	does not
77	$(15 - 6) \div 3 =$ ______ $\div$ ______ $=$ ______.	$9 \div 3 = 3$
78	Hence $(15 - 6) \div 3$ and $(15 \div 3) - (6 \div 3)$ are names for the same ______ number.	whole
79	In general if $(a - b) \div n$, $a \div n$, $b \div n$ all represent whole numbers, then $(a - b) \div n$ names the same whole number as $(\quad \div \quad) - (\quad \div \quad)$.	$(a \div n) - (b \div n)$

80	Hence division ______ (does, does not) distribute over subtraction from the right.	does
81	Division does not distribute over subtraction from the ______ (left, right). This fact may be demonstrated as it was for addition.	left

In this latter portion of the program some of the properties of division have been considered. In summary we note that order of performing operations and hence of notation is extremely important. An awareness of this now can minimize a great deal of confusion later.

CHAPTER 10

MULTIPLICATION TECHNIQUES

Multiplication techniques use the commutative, associative and distributive properties and the special properties of 0 and 1. In this chapter, no new concepts are introduced.

1 Consider the product 2×23.

$2 \times 23 = 2 \times [(2 \times 10) + \_\_\_\_]$. 3

2 By the distributive property this becomes

$[2 \times (2 \times 10)] + (2 \times \_\_\_\_)$. 3

3 Using the associative property

$[2 \times (2 \times 10)] + (2 \times 3) = [(2 \times 2) \times \_\_\_\_] + (2 \times 3)$ 10

$= (4 \times 10) + 6$

$= 46$.

4 Consider the product 6×14.

$6 \times 14 = 6 \times [(1 \times 10) + \_\_\_\_]$ 4

$= [6 \times (1 \times 10)] + (\_\_\_\_ \times 4)$. 6

5 Hence, $6 \times 14 = (6 \times 10) + \_\_\_\_$. 24

6 $(6 \times 10) + 24$ may be written as

$(6 \times 10) + [(2 \times \_\_\_\_) + 4] =$ 10

$[(6 \times 10) + (\_\_\_\_ \times 10)] + 4$. 2

7 Then $6 \times 14 = [(6 + 2) \times 10] + 4$

$= (\_\_\_\_ \times 10) + 4$ 8

$= 84$.

8	Consider 6×14 as $(6 \times 10) + (\underline{\quad\quad})$. Then $6 \times 14 = 60 + \underline{\quad\quad}$ $= \underline{\quad\quad}$.	(6×4) 24 84
9	Consider the product 43×12. $43 \times 12 = (40 + 3) \times 12$ $= (40 \times 12) + (3 \times \underline{\quad\quad})$	 12
10	Write 12 in expanded form and use the distributive property to obtain $[40 \times (10 + 2)] + [3 \times (10 + 2)] =$ $[(4 \times 10 \times 10) + (4 \times 10 \times 2)] + [(3 \times 10) + (\underline{\quad\quad})]$.	 (3×2)
11	And use the commutative, associative and distributive properties to obtain $(4 \times 100) + [(8 \times 10) + (3 \times 10)] + \underline{\quad\quad} =$ $(4 \times 100) + [(8 + 3) \times \underline{\quad\quad}] + 6$.	 6 10
12	Then $43 \times 12 = 400 + \underline{\quad\quad} + 6$ $= \underline{\quad\quad}$.	110 516
13	To find the product $\begin{array}{r} 43 \\ \underline{\times\ 12} \end{array}$ one thinks $(2 \times \underline{\quad\quad})$ and $(2 \times \underline{\quad\quad})$ which would give $\underline{\quad\quad} + \underline{\quad\quad} = 86$.	 3; 40 6; 80
14	Then, $(10 \times \underline{\quad\quad}) + (10 \times \underline{\quad\quad})$ giving $\underline{\quad\quad} + \underline{\quad\quad} = \underline{\quad\quad}$.	3; 40 30; 400; 430
15	It now follows that $12 \times 43 = 430 + 86 = \underline{\quad\quad}$.	 516

CHAPTER 11

DIVISION TECHNIQUES

In Chapter 9 it was observed that division as an operation may or may not yield a whole number. However, from $a \div b$, with $a > b$, one could obtain a whole number quotient $\underline{q}$ and a remainder $\underline{r}$ where $r < b$ or $r = 0$. Thus $a \div b$ can be expressed in equivalent form $a = (q \times b) + r$ where $r < b$ or $r = 0$, provided $b \neq 0$.

In this chapter no new concepts are introduced.

1	$39 \div 3 = (30 + 9) \div 3$ $= (30 \div 3) + \underline{(\qquad \div \qquad)}$ $= 10 + 3$ $= \underline{\qquad}.$	$(9 \div 3)$ 13
2	$39 \div 3$ can be written in the equivalent form $39 = (\underline{\qquad} \times 3) + 0.$	13
3	$40 \div 3$ is not a whole number since $40 = 39 + \underline{\qquad}$ and $(39 + 1) \div 3$ is not a whole number.	1
4	$40 \div 3$ can be written in the equivalent form $40 = \underline{(\qquad \times \qquad)} + 1.$	(13×3)
5	$40 \div 3$ yields a quotient of $\underline{\qquad}$ and a remainder of 1.	13
6	Using the form $40 = (q \times 3) + r$ $q = \underline{\qquad}$ and $r = \underline{\qquad}$.	13; 1
7	$97 \div 4$ written in the form $a = (q \times b) + r$ is $97 = \underline{(\qquad \times \qquad) + \qquad}.$	$(24 \times 4) + 1$

8 Consider the problem 575 + 23. This could be written as (230 + 230 + _____) + 23 = 115

(230 + 23) + (230 + 23) + (115 + _____) = 23

10 + 10 + 5 = 25.

9 To divide 600 by 23, write

(230 + 230 + 115 + 25) + 23 which would not yield a whole number. This division yields

(230 + 23) + (230 + 23) + (115 + 23) + (25 + 23)

with a remainder of _____. 2

10 And 600 + 23 written in a = (q × b) + r form would be 600 = (26 × _____) + 2. 23

600 + 23 could be put in vertical form by noting on the right the number of multiples of the divisor. Consider the following:

23 / 600

11 230 | _____ 10

370

12 230 | _____ 10

140

13 115 | _____ 5

25

14 23 | _____ 1

15 _____ is the quotient. 26

16 The remainder is _____. 2

Consider 15,119 ÷ 13. Writing this in vertical form would give

13 / 15119
17 13000 | ____ 1000
2119
18 1300 | ____ 100
819
19 650 | ____ 50
169
20 130 | ____ 10
39
21 39 | ____ 3

22 The quotient would be ____ and the remainder is ____. 1163 0

23 The mathematical sentence equivalent to 15,119 ÷ 13 is 15,119 = (____ × 13) + 0. 1163

CHAPTER 12

SENTENCES, NUMBER LINE

12-1. Introduction

Up to this point we have been using mathematical symbols for numbers, for operations, and for relations between numbers. One purpose of this chapter is to combine these symbols in certain ways to form number sentences. Formulas and equations are forms of number sentences.

1	7 - 5 = 2 is a ______ ______.	number sentence
2	7 - 5 = 9 also is a ______ ______.	number sentence
3	7 - 3 + 5 = 0 ______ (is, is not) a number sentence.	is
4	7 - 3 + 5 ______ (is, is not) a number sentence.	is not

5 Consider the symbols: 7, 5, 2, = . Indicate which of the following are number sentences.

☐ (a) 7 + 2 = 5

☐ (b) 7 = 2 + 5

☐ (c) 5 + 2 = 7

☐ (d) 7 + 5 + 2

5(a) Correct. This is a number sentence, even though it is not true. Refer to 5(b) and 5(c).

5(b) Correct, and also is a true number sentence. Refer to 5(a) and 5(c).

5(c) Correct, and also is a true number sentence. Refer to 5(a) and 5(b).

5(d) Incorrect. This is not a number sentence, but is a number phrase. See 5(a), 5(b), 5(c).

6	In the number sentence $2 + 3 > 7$, the symbol ______ acts as a verb.	$>$
7	In the number sentence $2 + 3 < 7$, the symbol $<$ acts as a ______.	verb
8	$7 + 2 = 9$ is a ______ (true, false) number sentence.	true
9	$7 + 2 = 10$ is a ______ (true, false) number sentence.	false
10	$7 + 2$ and $5 + 4$ are examples of number phrases. What symbol may be used as a verb to form a true number sentence from these two phrases? ______	$=$
11	What symbol may be used as a verb to form a false number sentence from the two phrases of Frame 10? ______	$\neq$ or $>$ or $<$

12-2. Open Number Phrases and Sentences

If a number phrase has a space not filled by a numeral, it is usual to fill that space by some letter such as n or a. For instance, $n + 7$ is a number phrase in which a numeral is represented by n. This phrase is called an open number phrase since the numeral to replace n is open to assignment. When one or more open number phrases are used in a number sentence, the sentence is called an open number sentence.

12 Which of the following are open number sentences?
(Check one or more.)

☐ (a) $3 + n = n + 3$

☐ (b) $5 + 2 = 9 - 2$

☐ (c) $5 > 2 + n$

☐ (d) $n + 8 = 16$

☐ (e) The sum of 8 and 7 is 19.

12(a) Correct. This number sentence is made up of two open number phrases; hence, it is an open number sentence. Note that 12(c) and 12(d) also are correct.

12(b) Incorrect. This is not an open number sentence. However, it is a true number sentence.

12(c) Correct. This number sentence is made up of one open number phrase. Note that 12(a) and 12(d) also are correct.

12(d) Correct. This number sentence is made up of one open number phrase. Note that 12(a) and 12(c) also are correct.

12(e) Incorrect. This number sentence is a false number sentence, and it is not open.

13 Which of the following number sentences are true, false, and/or open? (Check appropriate answers.)

True	False	Open	
☐	☐	☐	(a) $8 + 7 = 12$
☐	☐	☐	(b) $n + 4 = 4 + n$
☐	☐	☐	(c) $3 > 2 + 6$
☐	☐	☐	(d) $4 + 5 \neq 10 - 2$
☐	☐	☐	(e) $n + 8 = 16$

13(a) False, and is not open.

13(b) Open and True. This number sentence is made up of two open number phrases; hence it is an open number sentence. And, $n + 4 = 4 + n$ is true for all whole numbers n.

13(c) False. This is not a true number sentence, and it is not open.

13(d) True. This is a true sentence, and it is not open.

13(e) Open. We have no way of telling whether it is true or false. Also, see response 13(b).

12-3. Solving Open Sentences

From $A = \{0, 1, 2, 3, 4\}$ select the subsets of A such that each member of the subset selected makes the number sentence true in the following frames.

14	$n + 2 = 5$ ______	{3}
15	$n + 2 < 5$ ______	{0, 1, 2}
16	$n + 2 > 5$ ______	{4}

Each set selected above, provided it is a subset of A and provided each of its members makes the corresponding sentence true, is called the solution set for the open sentence.

17	Any member which makes an open sentence true is called a s______n of the open sentence.	solution
18	All the numbers, any one of which makes an open sentence true, form a ______ of solutions called the solution set of the open sentence.	set
19	Write the solution set of whole numbers for the number sentence $n - 4 < 5$. ______	{4, 5, 6, 7, 8}
20	When we have found the set of all solutions of an open sentence, we say that we have ______ the sentence.	solved

12-4. Use of Mathematical Sentences

21 Given the open sentence $n + 5 = 7$. Which of the following is represented by this open number sentence? (Check one.)

☐ (a) What number added to two equals seven?

☐ (b) John has seven pennies. He has five in one hand. How many does he have in the other hand?

☐ (c) Bill has five animals. He buys seven more. How many does he have all together?

21(a) Incorrect. There is no relationship between this statement and the open number sentence. This statement is represented by the number sentence $2 + n = 7$.

21(b) Correct.

21(c) Incorrect. An open number sentence to represent this situation is $5 + 7 = n$.

22 There are 22 children in a class. Ten of the children are boys. How many girls are there? Select the open sentences which express the relationship between the numbers involved. (Check all correct responses.)

☐ (a) $10 + n = 22$

☐ (b) $10 + 22 = n$

☐ (c) $22 - n = 10$

☐ (d) $n = 22 - 10$

22(a) Correct. This number sentence mathematically represents the problem which was given in words.

22(b) Incorrect. This number sentence is false since the set of girls has already been included in the 22 members of the class.

22(c) Correct. Read explanation for 22(a).

22(d) Correct. Read explanation for 22(a).

23 What is the solution of the problem in Frame 22? ______

12 girls

12-5. Solution Sets on the Number Line

Sometimes a number line is used to represent the solution set of a number sentence such as $7 - 4 = n$. See Figure 12.1 below.

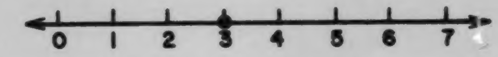
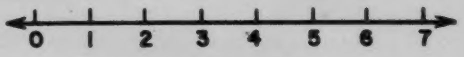


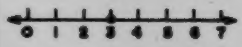
Figure 12.1

24 The solid "dot" on the number line in Figure 12.1 indicates the solution of the number sentence $7 - 4 = n$ is $n =$ ______.

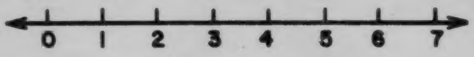
3

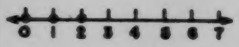
25 Using solid dots, indicate the solution set of the number sentence $n + 4 = 7$ on the number line below.



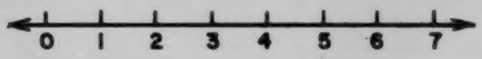


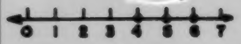
26 Using solid dots, indicate the solution set of the number sentence $n + 4 < 7$ on the number line below.





27 Using solid dots, indicate the solution set of the number sentence $n - 4 < 3$ on the number line below.





CHAPTER 13

AN INTRODUCTION TO GEOMETRY: POINTS, LINES AND PLANES

13-1. Points

In this chapter we discuss three basic ideas in geometry: points, lines and planes. We consider some of the properties of these familiar ideas and our discussions will not set up any formal deductive system. The first idea discussed is that of point. The term point is undefined in the study of geometry.

1 The symbol 2 is not a number but represents a ______.

number

2 In geometry, the symbol • is not a point, but may be used to represent a ______.

point

3 Select the better representation of a point: (Read all responses.)

☐ (a) The sharp end of a straight pin.

☐ (b) The head of a straight pin.

☐ (c) The location where two adjacent walls and the floor of a room meet.

☐ (d) The eraser end of a pencil.

3(a) If the pin is considered stationary, this response is correct. It is incorrect if the pin is thought of as moving.

3(b) This response includes many more points than the example in 3(a). Hence, it is incorrect.

3(c) This response is correct.

3(d) This response is incorrect. Even if the pencil is considered stationary, the eraser end of a pencil is a representation of many points.

4 Since we think of a point as a single location, does it move? ________ (yes, no) — no

5 The tip of the pencil you are using represents a point. Move the pencil to your other hand, then the tip represents ________________ (the same, a different) point. — a different

6 If a point P is represented by a small dot on a sheet of paper and the paper is moved, the ________ does not move. However, the representation of the point does move. — point

7 A dot on a sheet of paper can represent a point but actually it represents the following: (Check all correct responses.)

☐ (a) zero points

☐ (b) only one point

☐ (c) only two points

☐ (d) many points

7(a) This response is incorrect because any representation on paper covers many points.

7(b) This response is incorrect because any representation on paper covers many points.

7(c) This response in incorrect because any representation on paper covers many points.

7(d) This response is correct because any representation on paper covers many points.

Although it is not accurate, we use a small dot on a sheet of paper as a model of a point. It is a representation of a point, not the actual point.

Points generally are labeled by capital letters, such as A, B, P, Q and so forth.

In this sub-program we have stressed the following: a point involves position only; a point may be represented by a dot on paper, the end of some pointed object such as a needle or pencil, the corner of a room, or by a capital letter such as P. If the dot is erased or the needle is moved, the original point remains, since it is a position. Note that no formal definition has been given for a point. The term point is undefined in the development of geometry.

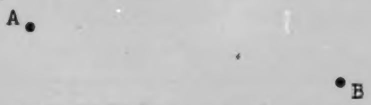
13-2. Sets of Points

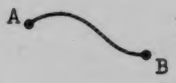
We think of lines, curves and planes as sets of points, and space as the set of all points. If space is the set of all points (or locations), we shall find that geometric figures such as lines, curves, rays, angles, triangles, and circles may be thought of as subsets of the set of all points in space.

8 Space is thought of as the set of all ______.

points or locations

9 Given points A and B below. Place your pencil on point A and, without lifting your pencil from the paper, move the pencil to point B.

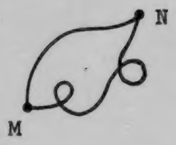




The path you made represents a set of ______.

points

10 Given points M and N below. Using a pencil make two different paths from M to N.



The paths represent ______________ (the same, different) sets of points.

different

11 A curve is a ______ of points all of which are on a particular path from a given point A to a given point B and including A and B. Frequently in other mathematical considerations this definition is modified to include curves which do not have endpoints. The line is a special case of a curve without endpoints.

set

12 The number of curves from given point A to given point B is:

☐ (a) only one

☐ (b) only two

☐ (c) any whole number

12(a) This is incorrect. It is possible to draw more than one.

12(b) This is incorrect. It is possible to draw more than two.

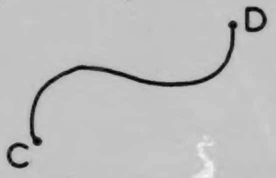
12(c) This is correct. In fact, there is no limit to the number which can be drawn.

13 The path from C to D is called:

☐ (a) a point

☐ (b) a curve

☐ (c) space



13(a) Incorrect. The path contains many points.

13(b) This is correct.

13(c) Incorrect. Space is the set of all points.

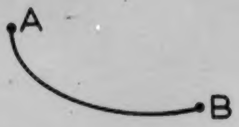
14 The number of curves from point P to point Q is ______________.
(finite, infinite)

infinite

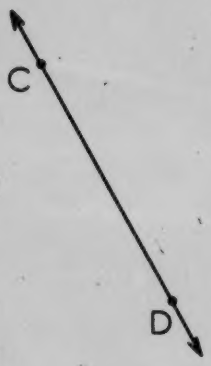
15	The set of points which make up each curve is ________. (finite, infinite)	infinite
16	If a path from A to B is made by moving a pencil along a ruler or a straight-edge, we refer to the path as a ______ segment.	line
17	We use the symbol $\overline{CD}$ to represent the line ______ determined by the points C and D. C and D are called endpoints and are a part of the line segment.	segment
18	Name the line segment (sometimes unnecessarily called a straight line segment) determined by the points A and B. ______	$\overline{AB}$
19	$\overline{MN}$ contains points M and ______.	N
20	If $\overline{AB}$ is extended in both directions so that it does not stop at any point, the result is a ______.	line
21	Arrows are used to indicate that a line does not stop. Thus $\overleftrightarrow{AB}$ is used as a symbol for the line containing the points ______ and ______.	A B
22	The line segment $\overline{AB}$ is a subset of the line ______.	$\overleftrightarrow{AB}$

23 Which of the following represents a special curve called a line?

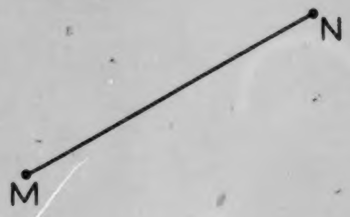
☐ (a)



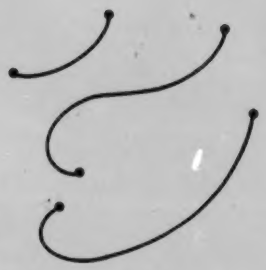
☐ (b)



☐ (c)



☐ (d)



23(a) Incorrect. It represents a curve, not a line.

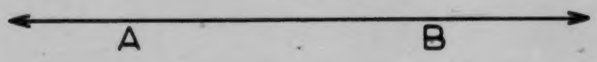
23(b) This is correct.

23(c) Incorrect. It represents a line segment, not a line.

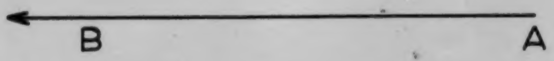
23(d) Incorrect. The picture consists of several curves.

24 Which of the following is a model of $\overleftrightarrow{AB}$ determined by the points A and B?

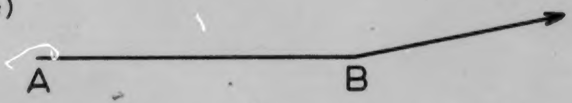
☐ (a)



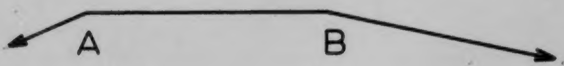
☐ (b)



☐ (c)



☐ (d)



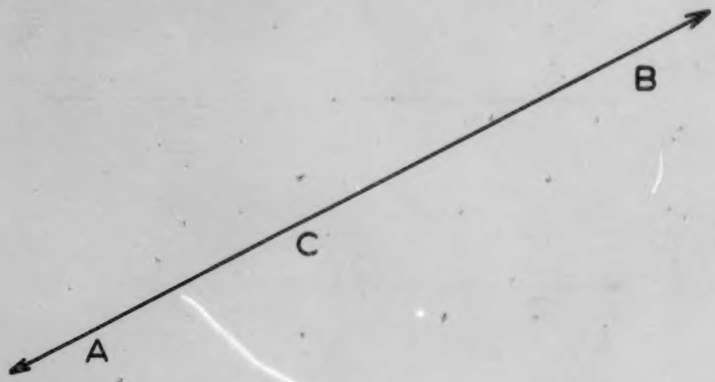
24(a) This continues in both directions through A and B. Correct.

24(b) This continues in only one direction through B. Incorrect.

24(c) Incorrect.

24(d) Incorrect.

25 Which of the following statements are true concerning the model below?



☐ (a) $\overleftrightarrow{AB} = \overleftrightarrow{BC}$

☐ (b) $\overline{CB} = \overline{AC}$

☐ (c) $\overline{AB} = \overline{BA}$

☐ (d) $\overline{AB} = \overleftrightarrow{AB}$

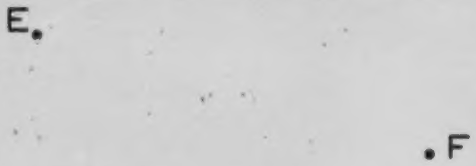
25(a) Every point on $\overleftrightarrow{AB}$ is a point on $\overleftrightarrow{BC}$ and conversely. Correct.

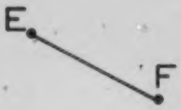
25(b) The points on $\overline{CB}$ are not the same as the points on $\overline{AC}$. Incorrect.

25(c) Every point on $\overline{AB}$ is a point on $\overline{BA}$ and conversely. Correct.

25(d) The points on $\overline{AB}$ are not the same as the points on $\overleftrightarrow{AB}$. $\overline{AB}$ is a part of $\overleftrightarrow{AB}$. Incorrect.

26 Draw a model of the line segment determined by E and F, the pair of points given below.

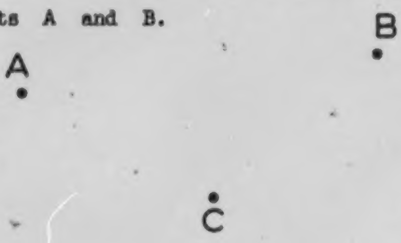


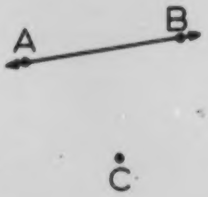


27 $\overline{EF}$ in Frame 26 is the set of all points on $\overleftrightarrow{EF}$ between E and F and includes the points E and ______.

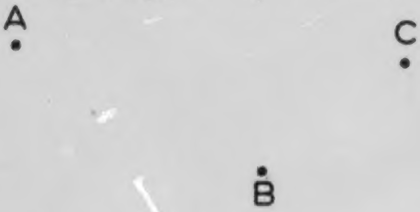
F

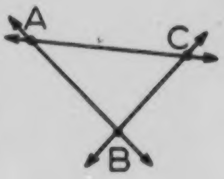
28 Draw a model of the line determined by the points A and B.



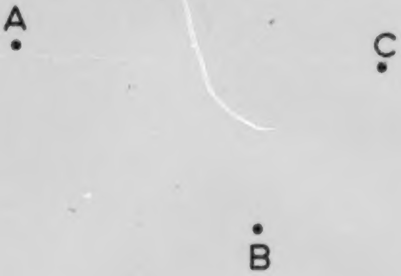


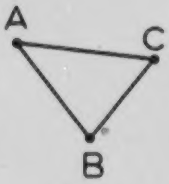
29 Draw representations of all possible lines determined by the points A, B and C.





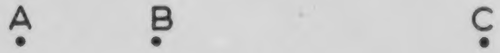
30 Draw a model of all possible line segments determined by the points A, B and C.

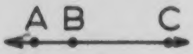




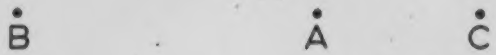
Henceforth, we will dispense with "a model of" or "a representation of" when you are asked to draw a model of a line or a line segment, with the understanding that it is possible to draw the model of or the representation of, but impossible to draw the line or line segment.

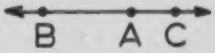
31 Draw all possible lines through all three points A, B and C.





32 Draw all possible lines through all three points A, B and C.





33 The line in Frame 32 may be denoted by

______

or ______

or ______

or ______

or ______

or ______

$\overleftrightarrow{BA}$

$\overleftrightarrow{AB}$

$\overleftrightarrow{BC}$

$\overleftrightarrow{CB}$

$\overleftrightarrow{AC}$

$\overleftrightarrow{CA}$

34 $\overleftrightarrow{AB}$ has the following number of endpoints: (Check one.)

☐ (a) zero ☐ (c) two

☐ (b) one ☐ (d) many

34(a) Correct. A line extends indefinitely in both directions and has no endpoints.

34(b) Incorrect. A line has no endpoints.

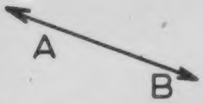
34(c) Incorrect. A line has no endpoints.

34(d) Incorrect. A line has no endpoints.

35 What is the greatest number of lines that can be drawn through a given point? ______

infinitely many

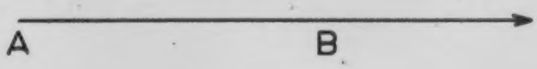
36 Draw all possible lines through both A and B.



37 How many lines can be drawn through any two different points? ______

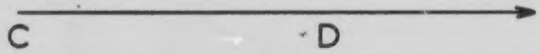
one and only one

38 The figure below is a model of the ray from A through B. The ray is denoted by $\overrightarrow{AB}$.



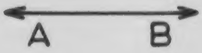
The figure below is a model of a ray and is denoted by ______.

$\overrightarrow{CD}$

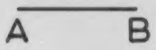


39 Given points A and B. Draw a line or portion of a line determined by A and B and having

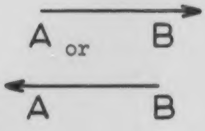
(a) no endpoints



(b) two endpoints



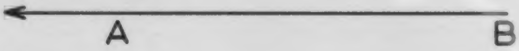
(c) one and only one endpoint



40 A ray is a portion of a line starting at a point and including that point, and extending in ______ direction(s).
(zero, one, two)

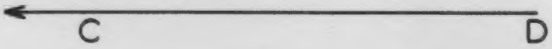
one

41 The figure below is a model of the ray with endpoint B and extending through A. The ray is denoted by $\overrightarrow{BA}$.

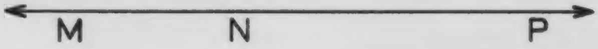


The figure below is a model of a ray and is denoted by ______.

$\overrightarrow{DC}$



42 Consider the following model:



Four different rays are:

______

$\overrightarrow{MN}$ or $\overrightarrow{MP}$
These are the same ray.

______

$\overrightarrow{PN}$ or $\overrightarrow{PM}$
These are the same ray.

______

$\overrightarrow{NP}$

______

$\overrightarrow{NM}$

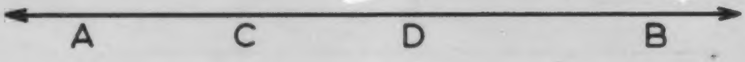
43 $\overrightarrow{AB}$ is a set of points consisting of A and all points of $\overleftrightarrow{AB}$ which are on the same side of A as the point ______.

B

44 Is the set of points $\overrightarrow{AB}$ the same as the set of points $\overrightarrow{BA}$? ______
(yes, no)

no

45 With reference to the figure below, which of the following statements are true?



☐ (a) $\overrightarrow{AC} = \overrightarrow{AB}$

☐ (b) $\overrightarrow{DA} = \overrightarrow{DC}$

☐ (c) $\overrightarrow{CA} = \overrightarrow{CB}$

☐ (d) $\overrightarrow{CD} = \overrightarrow{DB}$

45(a) Both rays consist of the same points, therefore they are equal. Correct.

45(b) Both rays consist of the same points, therefore they are equal. Correct.

45(c) These rays have the same endpoints but are in opposite directions. They are different rays, and therefore are not equal. Incorrect.

45(d) These rays have the same directions but have different endpoints, therefore they are not equal. Incorrect.

In this sub-program we have studied sets of points which are subsets of the set of all possible points or locations. Any path from one point to another point and including the two points is called a curve. We have devoted most of our attention to line segments and lines. A line segment is an infinite set of points, two of which are called its endpoints. A line is presented as the extension of a line segment extending continuously in both directions. A line also is an infinite set of points. We usually use the symbols $\overline{XY}$ and $\overleftrightarrow{XY}$ for the line segment and the line respectively, determined by the points X and Y. A line is a special case of a curve. A ray from B through C and extending continuously in the direction of C is denoted by $\overrightarrow{BC}$.

13-3. Planes and the Relationships of Points, Lines and Planes

A plane is a subset of the points in space. It is infinite in extent. In this sub-program some relationships between points, lines and planes are illustrated.

46	Think of your desk top as a model of a plane. This is only a r______n of a part of a plane.	representation
47	Such a model represents a set of ______ in space.	points
48	If the desk were moved, the ______ of points would not change.	set
49	The plane represented by the desk top is the set of all points obtained by extending all line ______ with endpoints on the desk top.	segments
50	Consider two points A and B on this model of a plane. A and B determine ______ (how many) lines?	one and only one
51	If A and B are different points in space, there is one and only one line passing through A and ______.	B
52	Property 1: Through any two ______ points in space there is exactly one line.	different
53	Consider two points A and B which lie in a model of a plane. $\overleftrightarrow{AB}$ ______ (does, does not) lie in the plane.	does
54	Property 2: If two different points lie in a plane, the line determined by the two points ______ in the plane.	lies
55	Select another model of a plane holding it above the desk. (This could be a small piece of cardboard.) The plane represented by the desk top and the plane represented by the cardboard model are ______ (the same, different).	different
56	Hold the cardboard model of a plane in one hand so that your thumb is at point B and your middle finger is at point A. Does $\overleftrightarrow{AB}$ lie on the plane? ______ (yes, no)	yes

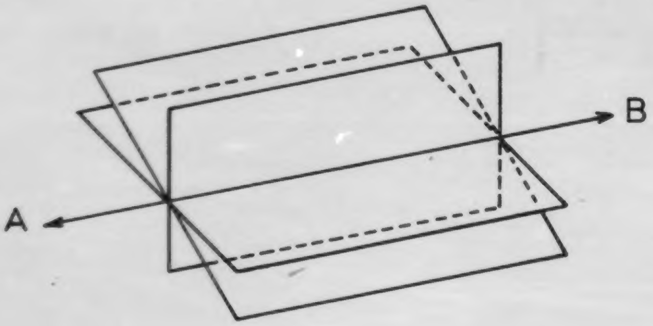
57 The number of planes which can be represented by the cardboard model holding points A and B stationary is: (Check one.)

☐ (a) one ☐ (b) two ☐ (c) many

57(a) Incorrect. See 57(c).

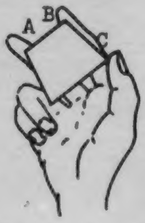
57(b) Incorrect. See 57(c).

57(c) Correct. As a matter of fact, there are infinitely many different planes containing $\overleftrightarrow{AB}$. See the picture below.



58 Property 3: Through two points in space, and hence through a line in space there are ________ possible planes. All of these planes intersect in in a line, that is, their intersection is a line.
(how many)

infinitely many

59	How many positions of the cardboard model are possible holding points A, B and C stationary? ______  	one and only one
60	Property 4: Any three points not on the same line determine one and only one ______.	plane
61	Consider $\overleftrightarrow{AM}$ on another model of a plane. Point M ______________ lie on $\overleftrightarrow{AM}$. (does, does not)	does
62	Place your pencil point on M. The point ______ lies on the line represented by the pencil.	M
63	M is a point lying on both $\overleftrightarrow{AM}$ and the line represented by the pencil. M is called the intersection of these two ______.	lines
64	Property 5: If two different lines in space intersect, their intersection is one ______.	point

65 If two distinct lines in space intersect, the number of points in the intersection is: (Check one.)

☐ (a) zero ☐ (c) two

☐ (b) one ☐ (d) many

65(a) Incorrect. If there are no points in the intersection, the two lines do not intersect.

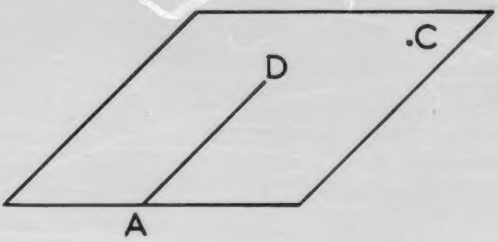
65(b) Correct.

65(c) Incorrect. If the intersection contains two points, the two lines are the same line and their intersection contains many points.

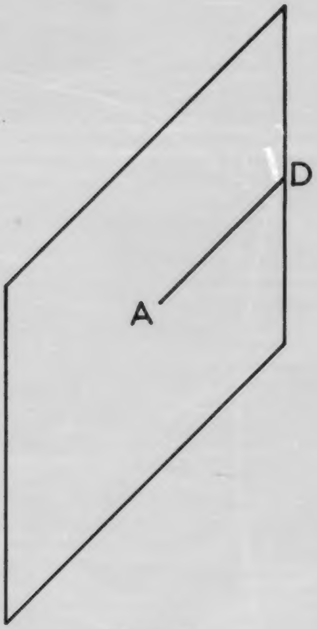
65(d) Incorrect. If the intersection contains many points, the two lines are the same line and their intersection contains many points.

66	Lay your pencil on a model of a plane so that a point of the pencil corresponds to point A in the model and another point of the pencil corresponds to point B in the model. The line represented by the pencil is denoted by ______.	$\overleftrightarrow{AB}$
67	The intersection of $\overleftrightarrow{AB}$ and the plane represented by the model is ______.	$\overleftrightarrow{AB}$
68	Property 6: If a line and a plane intersect, their intersection is either ______ point or the entire ______.	one line

For the following two frames, consider the pair of planes below, denoted by Plane 1 and Plane 2.

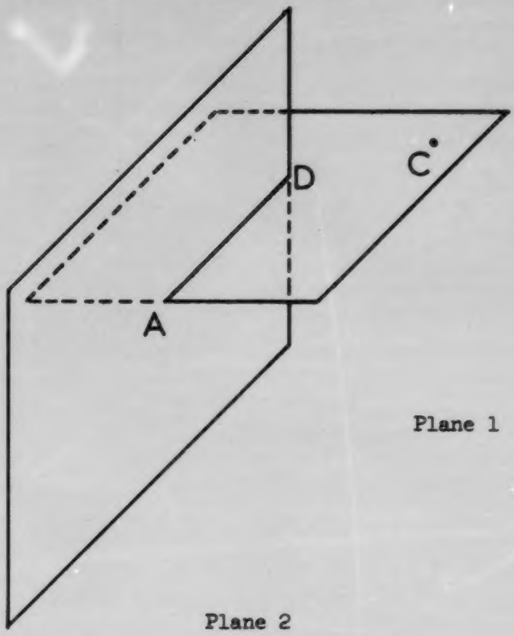


Plane 1



Plane 2

69 Hold a model of each plane so that points A and D are on both models and point C is on the model of Plane 1 but not on the model of Plane 2. See figure below.



The intersection of the model of Plane 1 and the model of Plane 2 is the line determined by the points _____ and _____.

A; D

70 Property 7: If two different planes intersect, their intersection is a straight ______.

line

In this sub-program we have discussed the meaning of a plane and the meanings of certain relationships between points, lines, and planes. From these relationships we have discovered several properties of points, lines, and planes. These properties are identified in the program and are facts we need to know as we continue the study of geometry. The seven properties are the following:

(1) Through any two different points in space there is exactly one line.

(2) If two different points lie in a plane, the line determined by the points lies in the plane.

(3) Through two points in space, and hence through a line in space, there are many possible planes.

(4) Any three points not on the same line determine one and only one plane.

(5) If two different lines in space intersect, their intersection is one point.

(6) If a line and a plane intersect, their intersection is either one point or the entire line.

(7) If two different planes intersect, their intersection is a line.

CHAPTER 14

CLOSED CURVES, POLYGONS AND ANGLES

14-1. Intersecting or Parallel Planes and Lines

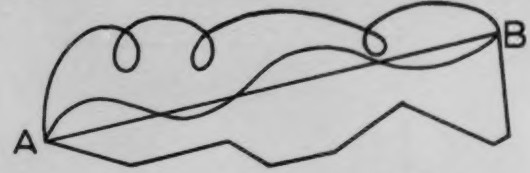
1	If two different planes ________ (do, do not) intersect, then they are parallel.	do not
2	If two different lines in the same plane do not intersect, then they are ______.	parallel
3	Two different planes either ______ or are parallel.	intersect
4	If a line and a plane do not intersect, then they are ______.	parallel

14-2. The Separation Properties of Planes, Lines and Points

5	Any plane in space separates all points in space into three subsets: the points on one side of the plane, the points on the other side of the plane, and the points in the ______ itself each form a subset.	plane
6	The set of points on one side of the p____e is called a half-space.	plane
7	And the set of points on the other side of the plane is called a ______.	half-space
8	The points on the separating plane do not lie in either ______.	half-space

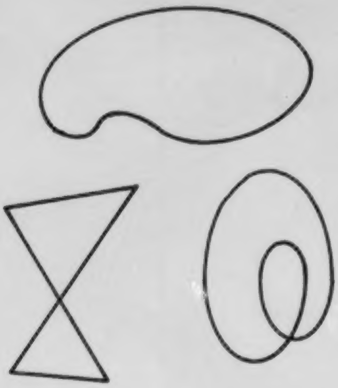
9	Any line in a plane separates all points of that plane into three subsets: the points on one side of the line, the points on the other side of the line, and the points in the ______ itself each form a subset.	line
10	The set of points in a plane on one side or the other side of a l______e in that plane is called a half-plane.	line
11	The set of points on the separating line does not lie in either ______.	half-plane
12	A point on a line separates all points of that line into three subsets: the points on one side of the point, the points on the other side of the point, and the ______ itself each form a subset.	point
13	The set of points on a line on one side or the other side of a p______t on that line is called a half-line.	point
14	The point of separation does not lie in either ______.	half-line
15	In Frames 5 - 13 we have considered the s______n properties by points, lines and planes.	separation

14-3. Plane Curves

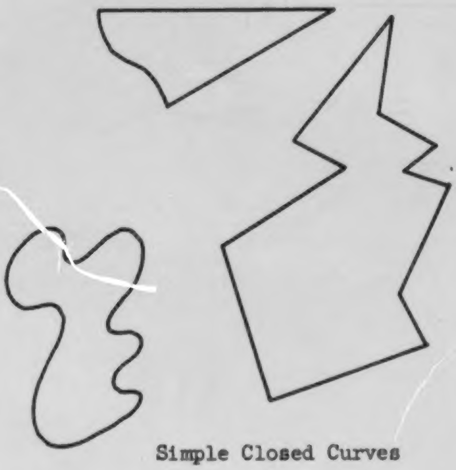
16	The set of all points which lie on a particular path from a given point A to a given point B and including A and B is called a ______.  Frequently in other mathematical considerations this definition is modified to include curves which do not have endpoints. The line is a special case of a curve having no endpoints.	curve
17	A line segment is a special case of a ______.	curve
18	The points of a curve always lie in the same plane. ______ (true, false)	false
19	If all the points of a curve lie in a plane, then the curve is called a plane ______.	curve
20	A ______ curve is a set of points which can be represented by a pencil drawing made without lifting the pencil off the paper.	plane
21	Line segments are examples of ______ curves.	plane

14-4. Closed Curves

A closed curve is a plane curve whose representation can be drawn without retracing and with the pencil point stopping at the same point from which it started. Some examples are:



Closed Curves



Simple Closed Curves

22	If a closed curve ______ (does, does not) intersect itself at any point, it is called a simple closed curve.	does not
23	We could speak of going around a simple ______ curve and, when we do, we pass through each point just once, except the starting point. (For the remainder of this chapter, we consider only simple closed curves in a plane.)	closed
24	Every simple closed curve in a plane separates the plane into three subsets: the interior points of the curve, the exterior points of the curve, and the ______ itself each form a subset.	curve

25	Any two points in the interior of a simple closed curve ______ (can, can not) be joined by a portion of a curve which does not intersect the original simple closed curve.	can
26	The same is true for any two points in the exterior of a simple closed curve. ______ (true, false)	true
27	The interior of any simple closed curve, together with the s_____e c_____d curve is called a region.	simple closed
28	The simple closed curve is called the boundary of the ______.	region
29	Note that the boundary of a region is a part of the ______.	region

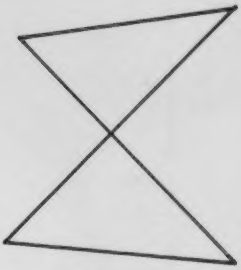
14-5. Polygons

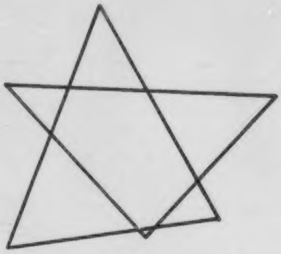
30	Polygons have special names according to the number of line segments involved. A triangle is a polygon with ______ line segments.	three
31	A quadrilateral is a simple closed curve made up of ______ (how many) line segments.	four
32	Pentagon, hexagon, octagon and decagon are names for ______ having 5, 6, 8 and 10 sides respectively.	polygons
33	If a simple closed curve in a plane is the union of three or more line segments, it is called a ______.	polygon

34 Note that the definition of polygon specifies _______ _______ curve in a plane. In other words, a curve can be the union of three or more segments, and not be a polygon. Polygons are defined as simple closed plane curves. Some plane curves are not polygons.

simple closed

Below are some plane curves which are not polygons.





14-6. Angles

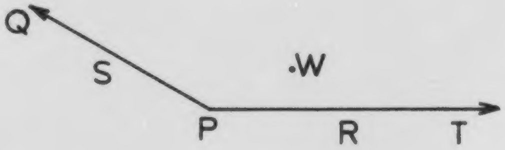
35 An angle is the union of two rays which have a common endpoint and ____________ (are, are not) parts of the same line.

are not

36 The common e_____t of the two rays is called the vertex of the angle.

endpoint

37 Note that in the figure below, S is a point of the angle QPT since S is on $\overrightarrow{PQ}$. Point W ____________ (is,is not) a point of the angle since it is not either $\overrightarrow{PQ}$ or $\overrightarrow{PT}$.



is not

38 The symbol for an ______ is ∠. — angle

39 An angle usually is denoted by naming three points of the angle: the first a point (not the vertex) on one ray, the second the ______, and the third a point (not the vertex) on the other ray. — vertex

40 Correct names for the angle represented by the figure below are: (Check all correct responses.)

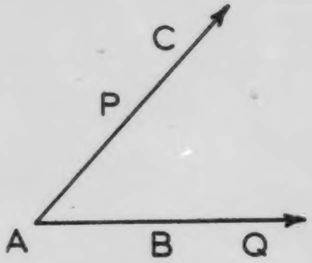
☐ (a) ∠CAB

☐ (b) angle PAQ

☐ (c) ∠BAC

☐ (d) angle QAP

☐ (e) ∠ABC



40(a) Correct. See also 40(b), 40(c), 40(d).

40(b) Correct. See also 40(a), 40(c), 40(d).

40(c) Correct. See also 40(a), 40(b), 40(d).

40(d) Correct. See also 40(a), 40(b), 40(c).

40(e) Incorrect, since the vertex A is written first.

41 In the preceding frame note that it is correct to say ∠CAB = angle PAQ = angle QAP since ∠CAB, angle PAQ and angle QAP are different ______ for the same angle. — names

We have been concerned in Chapter 13 and Chapter 14 with various geometric figures and some of their properties. We now turn to a consideration of their measures or sizes.

CHAPTER 15

METRIC PROPERTIES OF FIGURES

In Chapters 13 and 14, we considered lines, line segments, rays and simple closed plane curves. The latter included polygons with various numbers of sides as well as figures bounded by a curved line. Although we talked about the set of points interior to the simple closed curve, the set of points making up the boundary and the set of exterior points, we did not consider any of the properties of geometric figures which involve the idea of size or measure. We now consider some geometric figures to determine how the concepts of "is equal to," "is more than" and "is less than" apply.

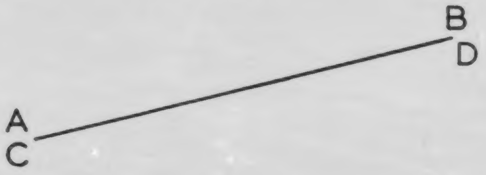
15-1. Congruence of Segments

1 $7 - 3 = 8 - 4$ is a true statement because $(7 - 3)$ and $(8 - 4)$ are different ______ for the number 4.

names

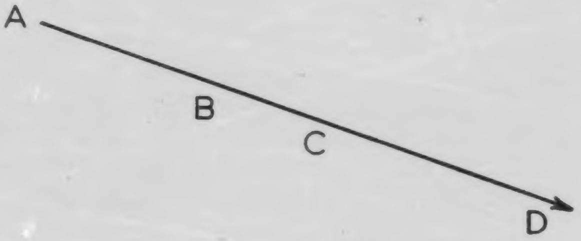
2 Consider the figure below. $\overline{AB} = \overline{CD} = \overline{AD} = \overline{CB}$ is a true statement because $\overline{AB}$, $\overline{CD}$, $\overline{AD}$ and $\overline{CB}$ are different names for the ______ set of points, namely, the line segment.

same



3 Consider the figure below. $\overline{AB}$ is not equal to $\overline{CD}$ because $\overline{AB}$ and $\overline{CD}$ do not ______ the same line segment.

represent



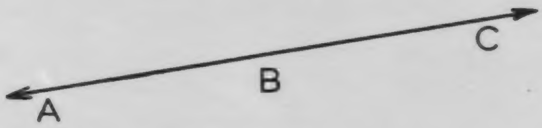
152

4 In the figure of Frame 3,

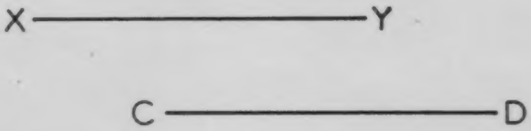
$\overrightarrow{AB}$ is _______________ to $\overrightarrow{AD}$. equal

(equal, not equal)

5 In the figure below, $\overleftrightarrow{AB}$ ____ $\overleftrightarrow{BC}$. =



6 In the figure below, $\overline{XY} \neq \overline{CD}$ is a true statement since $\overline{XY}$ and $\overline{CD}$ ______ different line segments. represent



7 Make a representation of the line segment $\overline{XY}$ of Frame 6 by placing one tip of the compass on X and adjusting it so the other tip falls on Y. If the compass is moved, the tips of the compass determine a line segment which is called a representation of the line segment ______. $\overline{XY}$

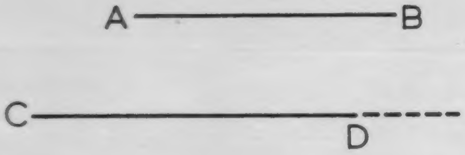
Compare the representation of $\overline{XY}$ with $\overline{CD}$ by placing one point of the compass on C and seeing if the other point of the compass falls on D. The other point of the compass ______________ fall on D. does

(does, does not)

8 Representations of line segments may be moved. Since a line segment is determined by its two endpoints and these points represent positions in space, they cannot be moved. Hence, line segments ____________ be moved. cannot

(can, cannot)

9 Given the line segments $\overline{AB}$ and $\overline{CD}$.



We may compare them by making a representation of $\overline{AB}$, say with a compass. If one point of the compass is placed on C, we find that the other point will fall ______________.
(between C and D, beyond D)

between C and D

10 Hence, we may say that $\overline{AB}$ is ______ $\overline{CD}$.

less than or
not as long as

11 We often express the relation in Frame 10 by using the symbol < . Thus, $\overline{AB}$ ____ $\overline{CD}$.

$\overline{AB} < \overline{CD}$

12 If any two line segments are compared by using representations of them, there are three possibilities:

Either $\overline{AB}$ ____ $\overline{CD}$,

or $\overline{AB}$ ____ $\overline{CD}$,

or $\overline{AB}$ and $\overline{CD}$ are congruent.

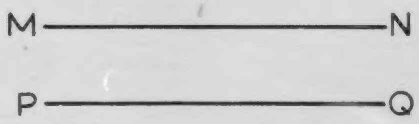
$\overline{AB} < \overline{CD}$

$\overline{AB} > \overline{CD}$

13 In the third possibility in Frame 12, namely, that $\overline{AB}$ and $\overline{CD}$ are congruent, we ______________ say $\overline{AB} = \overline{CD}$ because $\overline{AB}$ and $\overline{CD}$ represent different sets of points.
(can, cannot)

cannot

14 If one representation of a line segment is not more than and not less than the representation of another line segment, we use the word "congruent," denoted by the symbol $\cong$, rather than the word "equal," denoted by the symbol $=$. In the figure below, $\overline{MN}$ ______ $\overline{PQ}$.



$\overline{MN} \cong \overline{PQ}$

In this sub-program we have introduced the comparison of two or more line segments. This is basic to the idea of length to be discussed in the next chapter. In mathematics, since the word or symbol for "equal" is used only when two phrases name the same object, it is not possible to speak of equal line segments unless they do indeed represent exactly the same segment. Again, since a line segment is determined by two fixed points in space, it cannot be moved.

However, we do make a representation of a line segment by using a compass, a stretched string, marks, or a tracing on a card or paper, or a straightedge. These representations of line segments (we usually use a compass, if available, because it is handier) can be moved and for this reason we can compare two or more line segments. If we are given any two line segments such as $\overline{AB}$ and $\overline{CD}$, then by using representations of these segments we may discover that one and only one of the following statements is true:

$$\overline{AB} < \overline{CD}$$

$$\overline{AB} > \overline{CD}$$

$$\overline{AB} \cong \overline{CD}$$

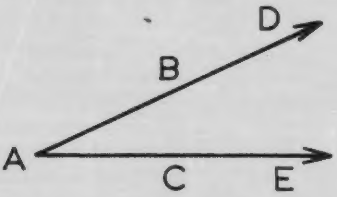
Notice in the last statement above we use $\cong$, the symbol for congruence to indicate that $\overline{AB}$ is not more than and is not less than $\overline{CD}$. The reason is that in the study of mathematics we want the word "equal" used if and only if we have different (or the same) names for the same thing. Note that $\overline{AB} \cong \overline{AB}$ is a true statement.

15-3. Comparison of Angles

In comparing angles we use tracings of the angles as we did in comparing segments.

16 In the figure below, the angles BAC and DAE are ______.

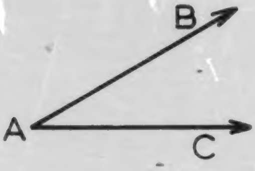
equal or =

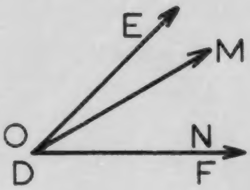


17 The reason we may use the equal symbol in Frame 16 is because "angle BAC" and "angle DAE" are ______ ______ for the same set of points.

different names

18 Given the angles BAC and EDF below.

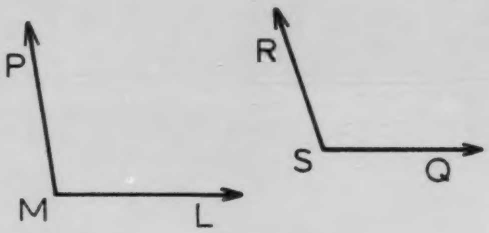




Make a tracing of ∠BAC, call it ∠MON, and place it over ∠EDF so that O falls on D and $\overrightarrow{ON}$ falls on $\overrightarrow{DF}$. Hence, ∠BAC ______ ∠EDF because $\overrightarrow{OM}$ falls within ∠EDF.

< or is less than

19 In a similar manner, compare $\angle RSQ$ with $\angle PML$.



The result of the comparison is $\angle RSQ$ ______ $\angle PML$. > or is more than

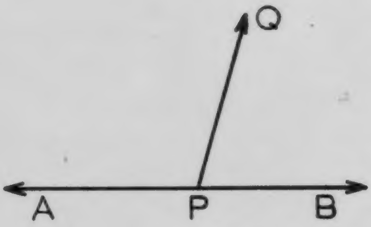
20 The three possibilities in comparing the angles MNO and RST are:

either $\angle MNO$ ______ $\angle RST$, >

or $\angle MNO$ ______ $\angle RST$, <

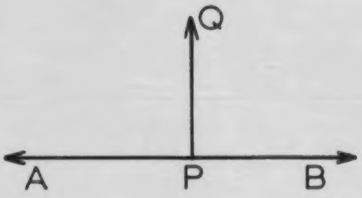
or $\angle MNO$ ______ $\angle RST$. $\cong$

21 Consider points A, P and B on the line $\overleftrightarrow{AB}$ such that P is between A and B.



Select a point Q not on $\overleftrightarrow{AB}$ and draw $\overrightarrow{PQ}$. By looking at the figure or making a tracing representation of $\angle APQ$, we find that $\angle APQ$ ______ $\angle BPQ$. >

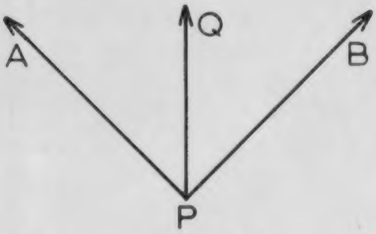
22 Use the figure below to compare $\angle APQ$ and $\angle BPQ$.



We find that $\angle APQ$ ______ $\angle BPQ$. ≅

If this is true, we call the angles APQ and BPQ right angles. Later we learn that the degree measure of each angle is 90°. Furthermore, if P is a point on $\overline{AB}$ and $\angle APQ \cong \angle BPQ$, we say that $\overrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{AB}$.

23 In the figure below, compare the two angles APQ and BPQ.



We find $\angle APQ$ ______ $\angle BPQ$. ≅

24 If $\angle MNO > \angle PRQ$ and $\angle PRQ > \angle ABC$, then it follows that $\angle MNO$ ______ $\angle ABC$. >

In this sub-program we have developed some ideas analogous to those for line segments. We may compare angles by using a representation of one of the angles, usually traced on paper, then placing the tracing over the other angle. Given any two angles such as $\angle ABC$ and $\angle DEF$, then by using representations of them we may discover that one and only one of the following statements is true:

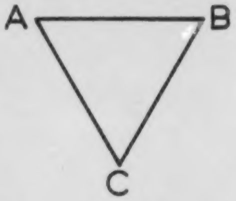
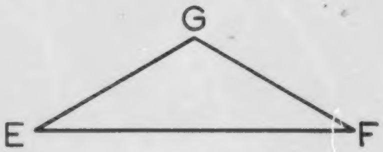
$$\angle ABC > \angle DEF$$

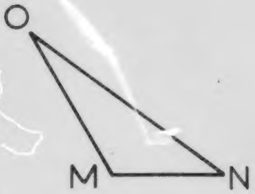
$$\angle ABC < \angle DEF$$

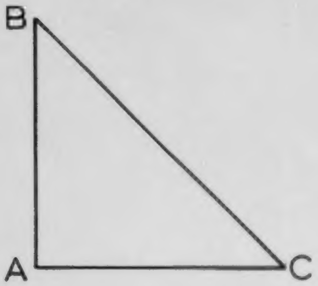
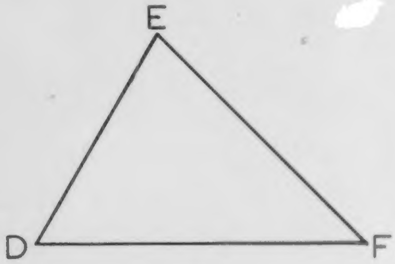
$$\angle ABC \cong \angle DEF$$

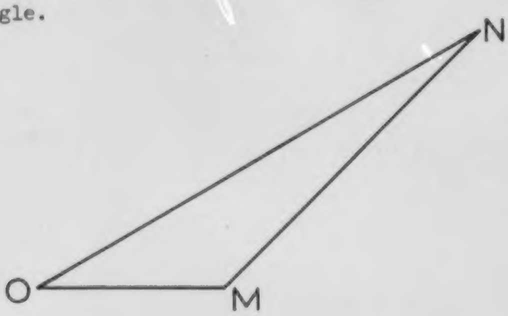
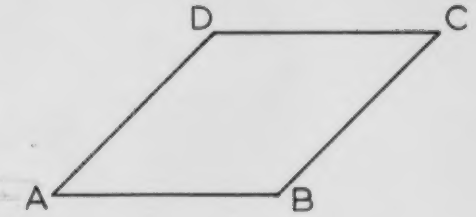
Right angles were discussed briefly and defined. They will be used more and more in succeeding chapters. One way to make a model of four right angles is to use a sheet of paper; fold and crease it once anywhere; then make a second fold so the crease folds on itself. The result is a model of four right angles.

15-4. Classification of Triangles and Quadrilaterals

25	In the triangle ABC, use a compass to compare the line segments $\overline{AC}$ and $\overline{BC}$ and the line segments $\overline{AB}$ and $\overline{BC}$. It is found that $\overline{AC}$ ______ $\overline{BC}$ and $\overline{AB}$ ______ $\overline{BC}$. 	$\overline{AC} \cong \overline{BC}$ $\overline{AB} \cong \overline{BC}$
26	From the statements in the response to Frame 25 $\overline{AC}$ ______ $\overline{AB}$.	$\overline{AC} \cong \overline{AB}$
27	If all three sides of a triangle are ______, then the triangle is called an equilateral triangle.	congruent
28	In the triangle EGF below, compare the sides $\overline{EG}$ and $\overline{GF}$ and the sides $\overline{EG}$ and $\overline{EF}$.  The results are $\overline{EG}$ ______ $\overline{GF}$ and $\overline{EG}$ ______ $\overline{EF}$.	$\overline{EG} \cong \overline{GF}$; $\overline{EG} < \overline{EF}$

29	From the statements in the response to Frame 28, $\overline{GF}$ ______ $\overline{EF}$.	$\overline{GF} < \overline{EF}$
30	In the triangle EGF of Frame 28, $\overline{EG}$ and $\overline{EF}$ are ______. Such a triangle is isosceles, that is, if two sides of a triangle are congruent, then the triangle is called an isosceles triangle.	≅ or congruent
31	In an isosceles triangle at least two sides are ______.	≅ or congruent
32	In an equilateral triangle all three sides are ______.	≅ or congruent
33	Triangle ABC in Frame 25 is both ______ and isosceles.	equilateral
34	In the triangle MNO, compare the side $\overline{MN}$ with the side $\overline{MO}$, and compare the side $\overline{MO}$ with the side $\overline{ON}$. The results are $\overline{MN}$ ______ $\overline{MO}$ and $\overline{MO}$ ______ $\overline{ON}$.  Hence, $\overline{MN} <$ ______ $< \overline{ON}$.	$\overline{MN} < \overline{MO}$; $\overline{MO} < \overline{ON}$ $\overline{MO}$
35	In the triangle MNO of Frame 34, no two ______ are congruent. Such a triangle is called a scalene triangle.	sides
36	In a scalene triangle no two sides are ______.	≅ or congruent
37	A triangle with all three sides congruent is ______.	equilateral
38	A triangle with at least two sides congruent is ______.	isosceles
39	A triangle with no two sides congruent is ______.	scalene

40	In the triangle ABC, $\overline{AB}$ is perpendicular to $\overline{AC}$. Hence, angle BAC is a ______ angle.  	right
41	A triangle which has a ______ angle is a right triangle.	right
42	The triangle ABC in Frame 40 is a ______ triangle.	right
43	Compare the sides $\overline{AB}$ and $\overline{AC}$ of triangle ABC in Frame 40. The result is $\overline{AB}$ ______ $\overline{AC}$.	$\overline{AB} \cong \overline{AC}$
44	From the answer in Frame 43, the triangle ABC is also ______.	isosceles
45	In the triangle DEF below, each of the angles is smaller than a ______ angle and each is called an acute angle.  	right

46	Since all angles of the triangle DEF of Frame 45 are ______ angles, the triangle is called an acute triangle.	acute
47	In the triangle MNO below, the angle OMN is greater than a ______ angle and is called an obtuse angle.  	right
48	The triangle MNO in Frame 47 is called an ______ triangle.	obtuse
49	Use a compass to compare the sides of the quadrilateral ABCD.  Each side is ______ to every other side.	congruent or ≅
50	A quadrilateral with all sides ______ is equilateral. Such a quadrilateral also is called a rhombus.	congruent
51	A quadrilateral with no pair of sides ______ is called a scalene quadrilateral.	congruent
52	A quadrilateral with all of its angles ______ angles is called a rectangle.	right

53	A quadrilateral with each pair of opposite sides congruent and ______ is called a parallelogram.	parallel
54	A rectangle also is a p______m.	parallelogram
55	An equilateral quadrilateral also is a p______m.	parallelogram
56	A quadrilateral with each side ______ to every other side and all its angles right angles is called a square.	congruent

In this sub-program we have discussed some of the classifications of triangles and quadrilaterals. These figures may be classified according to properties of their sides or according to properties of their angles. The most important of these are summarized as the following:

Triangles are	If
Equilateral	all three sides are congruent.
Isosceles	at least two sides are congruent.
Scalene	no two sides are congruent.
Right	one angle is a right angle.
Acute	each angle is less than a right angle.
Obtuse	one angle is an obtuse angle.

These classifications overlap. An equilateral triangle is also isosceles and acute. Right, acute, and obtuse triangles may also be isosceles. An acute triangle may be equilateral, isosceles, or scalene.

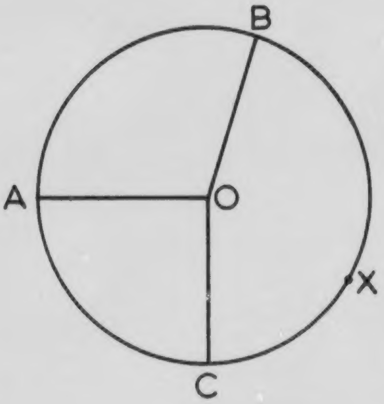
Quadrilaterals are	If
Equilateral	each side is congruent to every other side.
Scalene	no two sides are congruent.
Square	each side is congruent to every other side and each angle is a right angle.
Rectangle	all of its angles are right angles.
Parallelogram	each pair of opposite sides are congruent and parallel.

We now shift our attention to another simple closed curve, the circle. This curve bears certain relations to polygons of many sides.

15-5. Circles

57 Given the figure below. Use a compass to find a representation of the line segment $\overline{OA}$ and compare this segment with $\overline{OB}$ and $\overline{OC}$. The result is $\overline{OA}$ ______ $\overline{OB}$ ______ $\overline{OC}$.

$\overline{OA} \cong \overline{OB} \cong \overline{OC}$

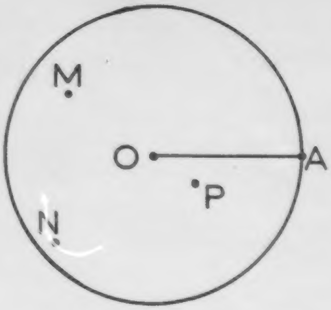


58 Select another point X on the simple closed curve in Frame 57 and compare $\overline{OX}$ with $\overline{OA}$. Since $\overline{OX}$ is ______ to $\overline{GA}$, it also is congruent to $\overline{OB}$ and to $\overline{OC}$.

congruent

Definition: A circle is a simple closed curve having a point O in its interior such that, if A and B are any two points on the curve, then $\overline{OA} \cong \overline{OB}$. $\overline{OA}$ and $\overline{OB}$ and all other line segments from O to points on the closed curve are called radii of the circle. The point O is called the center of the circle.

59 Given the circle with center 0 and radius $\overline{OA}$.



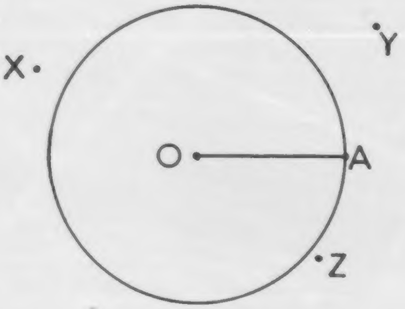
Select points M, N, P in the interior of the circle. Use a compass and compare $\overline{OM}$, $\overline{ON}$ and $\overline{OP}$ with the radius $\overline{OA}$. The results are:

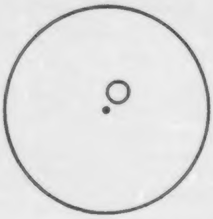
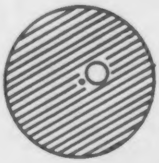
$\overline{OM}$ ____ $\overline{OA}$	$\overline{OM} < \overline{OA}$
$\overline{ON}$ ____ $\overline{OA}$	$\overline{ON} < \overline{OA}$
$\overline{OP}$ ____ $\overline{OA}$	$\overline{OP} < \overline{OA}$

60 A point such as M is in the ______ of the circle with center 0 and radius $\overline{OA}$ if $\overline{OM} < \overline{OA}$. interior

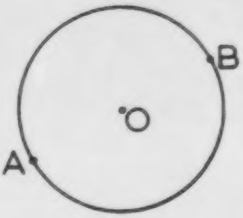
61 In another circle select the points X, Y, Z in the exterior of the circle and compare $\overline{OX}$, $\overline{OY}$ and $\overline{OZ}$ with $\overline{OA}$. The results are:

$\overline{OX}$ ____ $\overline{OA}$	$\overline{OX} > \overline{OA}$
$\overline{OY}$ ____ $\overline{OA}$	$\overline{OY} > \overline{OA}$
$\overline{OZ}$ ____ $\overline{OA}$	$\overline{OZ} > \overline{OA}$

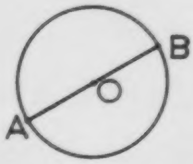


62	A point such as X is in the exterior of the circle if $\overline{OX}$ ____ $\overline{OA}$.	$\overline{OX} > \overline{OA}$
63	A circle is a simple closed curve. The interior of a circle with center O and radius $\overline{OA}$ consists of all points M such that $\overline{OM}$ ____ $\overline{OA}$.	$\overline{OM} < \overline{OA}$
64	The exterior of a circle with center O and radius $\overline{OA}$ consists of all points X such that $\overline{OX}$ ____ $\overline{OA}$.	$\overline{OX} > \overline{OA}$
65	Shade lightly the interior of the circle, center O. The union of the circle and its interior is called a circular ______.	region
		
66	The ______ is called the boundary of the circular region.	circle
67	A circular region is the ______ of a circle and its interior.	union

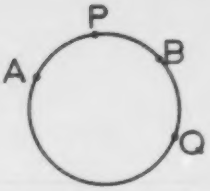
68 In a given circle, draw a radius $\overline{OA}$ and extend $\overline{OA}$ through O until it meets the circle on the other side of O at B. The line ______ $\overline{AB}$ is called a diameter of the circle.



segment



69 Consider a circle and points A and B on the circle. One portion of the circle (an arc) is written $\widehat{APB}$. Points A and B separate the circle into arcs ______ and ______.

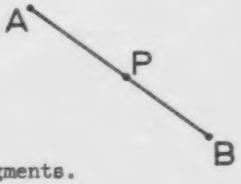


$\widehat{APB}$; $\widehat{AQB}$

70 A single point P on a circle ______ (does, does not) separate the circle into two parts.

does not

71 Given a line segment $\overline{AB}$. Point P between A and B separates the segment into ______ (how many) parts or line segments.



two

72 Two points on a circle separate the circle into ______ (how many) distinct parts or arcs.

two

73	Two arcs of a circle may or may not be congruent. If the points A and B are endpoints of a diameter, then the two parts of the circle are _______ arcs. Each of these arcs is called a semi-circle and $\overline{AB}$ is called the diameter of the semi-circle.	congruent
74	The union of a semi-circle and its diameter and its interior points is called a semi-circular _______.	region

In this sub-program we have introduced the concept of a circle as a particular simple closed curve, one that has a point O in its interior such that all line segments from O to points on the curve are congruent segments. A circle separates the plane into three regions: the interior of the circle, the exterior of the circle, and the circle itself. The union of the interior of a circle and the circle itself is called a circular region. Thus, the word "circle" refers to the boundary between the interior and the exterior of a circle. A circle is a set of points and is an example of a simple closed curve. Later we measure the length of a circle, but the "area of a circle" has no meaning. However, the "area of a circular region" does have meaning. When we talk about area, we mean the area of a circular region.

15-6. Summary

In this chapter we have used metric properties of sets of points, but we have not done any measuring nor have we defined measurement. We re-emphasize the fundamental meaning of the word "equal" and the symbol = as used only when we have two names for the same object. For line segments as well as other geometric figures, we try to obtain a representation of the line segment, such as the separation of the points of a compass. These representations can be moved, but the points and segments themselves cannot be moved.

In this chapter, by means of representations, we compare two line segments and all we are able to do is to say that one segment is shorter or longer than the other or that they are congruent. Hence, given two line segments $\overline{AB}$ and $\overline{MN}$, we say that one and only one of the following statements is true:

$$\overline{AB} < \overline{MN}$$

$$\overline{AB} > \overline{MN}$$

$$\overline{AB} \cong \overline{MN}$$

With angles we do the same thing. By comparison, we say that one and only one of the following statements is true:

$$\angle AOB < \angle MNR$$

$$\angle AOB > \angle MNR$$

$$\angle AOB \cong \angle MNR$$

Triangles and quadrilaterals are classified according to whether sides are congruent or greater than or less than.

A circle was discussed as a simple closed plane curve with a center and equal radii. The set of points interior to a circle together with the points on the circle form a set called a circular region.

CHAPTER 16

LINEAR AND ANGULAR MEASURE

16-1. Introduction

Much of our previous efforts have been directed to counting the members of a set and using these numbers in arithmetical operations of addition, subtraction, multiplication and division. The concept of measure also depends on counting.

1	If we wish to find out how many books are on a shelf, we ______ the number of books there.	count
2	On the other hand, we ______ the length of a desk.	measure

The act of measuring a line segment involves the selection of a suitable unit and applying this unit to the line segment, counting the approximate number of times this unit fits the line segment. This number of units is the length, or magnitude, of the line segment and is written as "n" units.

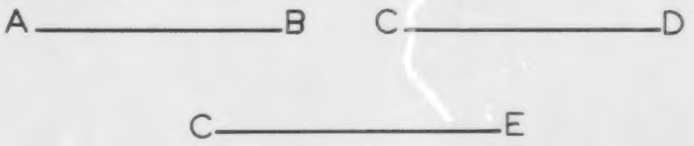
3	In measuring a line segment, we ______ the number of times that the unit is applied to the line.	count
4	Two line segments are equal if they consist of the same ______ of points.	set
5	The measure of two equal line segments is ________________. (the same, different)	the same

Our interest also lies in line segments which have the same measure but are not equal. Such line segments are said to be congruent. If $\overline{AB}$ and $\overline{CD}$ are congruent, we write $\overline{AB} \cong \overline{CD}$.

If two line segments are not congruent, then the measure of one is less than the measure of the other and we write $m(\overline{AB}) < m(\overline{CD})$ or $m(\overline{CD}) < m(\overline{AB})$ where $m(\overline{AB})$ denotes the measure of $\overline{AB}$ and $m(\overline{CD})$ denotes the measure of $\overline{CD}$.

6	The symbol for congruent is ______.	$\cong$
7	$\overline{AB} \cong \overline{CD}$ means that the line segments $\overline{AB}$ and $\overline{CD}$ have the ______ measure.	same
8	If two line segments are congruent, these line segments have the same ______, and also have the same length.	measure

To determine whether or not two line segments are congruent, it is necessary to compare their lengths. This can be done by placing the ends (tips) of a compass (or dividers) on the endpoints of one line segment and seeing whether the tips of the compass will coincide with the endpoints of the other line segment.



9	Set a compass to the length of $\overline{AB}$, compare to $\overline{CD}$ by placing one tip on C. Since the second tip falls on D, the line segments are ______.	congruent
10	Now consider another line segment $\overline{CE}$. Keeping the compass set to the length of $\overline{AB}$, place one tip of the compass on C and compare $\overline{CE}$ to the length of $\overline{AB}$. When the second tip falls between C and E, we say that the measure of $\overline{AB}$ is less than the measure of $\overline{CE}$ and write $m(\overline{AB})$ ____ $m(\overline{CE})$.	$<$
11	The measure of $\overline{CE}$ also is greater than the measure of $\overline{AB}$. This may be written ______ $> m(\overline{AB})$.	$m(\overline{CE}) > m(\overline{AB})$

One speaks of $\overline{AB}$ as less than $\overline{CE}$ if $m(\overline{AB}) < m(\overline{CE})$. For notational simplicity, this relationship is symbolized by writing $\overline{AB} < \overline{CE}$. In subsequent parts of this program, this simplified notation is used.

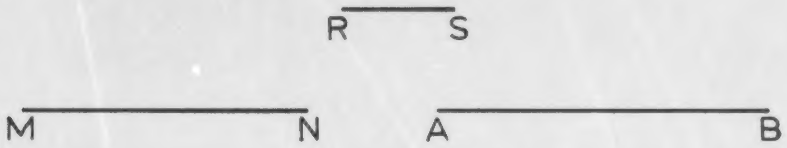
16-2. Measuring a Line Segment

Fundamentally, the measurement of line segments is a process of comparing one line segment to another one. If, however, the line segments are not congruent, it is not sufficient to determine whether one is greater than or less than the other.

One needs to select a line segment, for example $\overline{AB}$, to serve as a unit. The unit selected is arbitrary, and its measure is one. Once a unit is selected for a given problem, however, it is the smallest length that will be accepted in measurements of that problem. One may select as a unit any suitable length, for example one-third of an inch, as the arbitrary unit.

12	In comparing our unit $\overline{AB}$ to some line segment, there are three possibilities: the line segment may be ______ to, greater than, or ______ ______ the unit $\overline{AB}$.	congruent less than
13	If $\overline{CD} \cong \overline{AB}$, we say that $\overline{AB}$ and $\overline{CD}$ have the ______ ______.	same length or same measure
14	Since $\overline{CD} \cong \overline{AB}$, we may use either ______ or $\overline{AB}$ as a unit.	$\overline{CD}$
15	The number assigned to the unit $\overline{AB}$ is ______.	1 or one
16	The exact length of the ______ used in measurement is arbitrary.	unit
17	Any line segment which is congruent to our unit $\overline{AB}$ has a measure of ______.	1 or one

Select a line segment such as $\overline{RS}$ below as the arbitrary unit.



18	To measure a line segment such as $\overline{AB}$ for example and using $\overline{RS}$ as a unit, proceed as follows: set the compass such that its points are on R and S and strike off on $\overline{AB}$ a unit length from A; the measure from A to this point is one ______.	unit
19	From the endpoint of the unit segment now marked on $\overline{AB}$, strike off another unit length. The length from A to this second point is ______ ______.	2 units
20	On doing this a third time, if the final mark is at B, the length of $\overline{AB}$ is ______ ______.	3 units
21	In general, to measure a line segment, we first choose a suitable ______ and assign to it the measure one or 1.	unit
22	Then, with a compass, we strike off successive line segments each ______ to the unit.	congruent
23	If, on the last strike, the point of the compass coincides with the second endpoint of the line segment being measured, we count the number of units. If the count is n, then the length of the line segment is said to be ______ ______.	n units (n is a counting number)
24	The measure of the line segment $\overline{AB}$ in Frames 18 - 20 is ______.	3

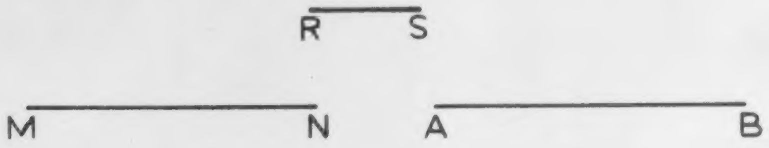
25	To determine the length of a line segment, one also must specify the ______.	unit
26	The length of a line segment $\overline{AB}$ used in Frames 18 - 20 is ______ ______.	3 units
27	To measure a line segment, ______ the number of times the chosen unit is used to completely cover the line segment.	count
28	If n units are used to measure a line segment, one says that the length of the line segment is ______ ______.	n units
29	The measure of the line segment in Frame 28 is ______.	n
30	Instead of using a unit as measured by a compass, select a starting point at or near one end of a straight-edge. Having selected a unit, mark off a unit length from the starting point. The measure of this segment is ______.	one or 1
31	From this point mark off another unit. The measurement from the starting point to this second endpoint is ______. In a similar manner additional units are marked off on the straight-edge.	two or 2

A ruler is formed by counting and labeling the units along a straight-edge.

32	The measure of a line segment ______ (does, does not) depend upon the endpoint from which the units are counted.	does not

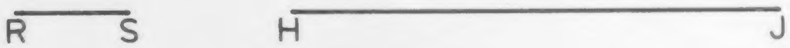
16-3. The Approximate Nature of Measure

$\overline{AB}$ measured in terms of the unit $\overline{RS}$ of the last section has a length of 3 units. We know, however, that other line segments will not necessarily be measured evenly with a given unit.



33	The sentence, "A line segment is 3 units long," means that the measure of the line segment is the number ______.	3 or three
34	When we measure $\overline{AB}$ and $\overline{MN}$ using $\overline{RS}$ as the unit, we find that each is ______ units long.	3
35	However, $\overline{AB}$ and $\overline{MN}$ are not ______.	congruent
36	Thus, to say that a line segment is 3 units long means that its length is closer to three units than to ______ units or to ______ units. In general, measurement of a line segment is approximate.	2; 4

Once a unit is selected for a given problem, it is the smallest length that will be accepted in the measurements of that problem.



37 When $\overline{RS}$ above is the unit, the measure of $\overline{HJ}$ lies half-way between 4 and 5. The line segment $\overline{HJ}$ should be assigned the measure

☐(a) $4\frac{1}{2}$ ☐(b) 4 ☐(c) 5

37(a) This response is incorrect. Once a unit is selected, the length must be specified as a whole number of these units, and the measure is that whole number.

37(b) This response is incorrect. 37(c) is correct. Remember that at best, measurement is an approximation. Here we have a difficult situation. To do better we would have to select a new unit, one that may be smaller than $\overline{RS}$. However, by convention, if a length is as much or more than one-half unit more than the smaller measure, we use the larger measure.

37(c) This response is correct. Read 37(b).

16-4. Standard Units

The actual selection of a unit is arbitrary. However, to give measurement meaning, it becomes necessary to agree upon a unit to be used universally. Such units are called standard units. The standard unit is one accepted by all concerned with its use so that a measurement may be interpreted at another time and place.

Historically, there have been many units. In the United States the National Bureau of Standards is responsible for the establishment of our standard units.

Today, there are two major systems of units in use. The British-American system has as basic units the familiar inch, foot, yard and mile. The particular unit depends upon the use to which the measurement will be applied. The second system in common usage is the metric system with the meter as the basic unit. In 1960, the meter was redefined in terms of the wave length of orange light from krypton 86.

16-5. Grouping of Units

The standard units are frequently converted in reporting ordinary lengths. For example, 12 inches to the foot, 3 feet to the yard, 5,280 feet to the mile and so forth. These equivalences are advantageous in specifying lengths.

38 For example, 41 inches may be written as
______ feet and ______ inches. 3; 5

39 It also may be written as ______ yard and 1
______ inches. 5

40 If a line segment has a length of 36 inches,
it is implied that the unit is the inch and
its measure is closer to ______ than to 35 36
or to ______. 37

41 By converting to feet the 36 inches become
______ feet. 3

42 But, if the unit is a foot, this implies that the
length of the line segment is closer to 3 feet
than to ______ feet or to 4 feet. 2

43 The assumed error in any quoted measure is at most
half a unit. In Frame 42, six inches is the
implied error. Therefore, a length of 3 feet
implies that its actual length lies between 30
inches and ______ inches. 42

44 This statement tells us much less than the original
statement that the line segment has a length of 36
inches. To correct this we can say that the line
segment is 3 feet, 0 inches long or 3 feet
long to the ______ inch. nearest

45 In a similar manner we could have said that the line
segment has a length of ______ yard to the nearest one
inch.

16-6. Summary

In measuring a line segment, one first must select a unit the length of which is assigned the number one. The unit is then applied to the line segment to be measured. Finally the number of successive segments, each congruent to the unit, is determined as the measure. If the last unit ends on the terminal point of the line segment, we have a measurement that is exact (to within experimental error). Usually it is necessary to determine the number of units that most nearly approximates the length of the line segment.

16-7. The Naming of Units

In the British-American system of standard units, the inch is frequently used. However, there are many occasions for which a smaller unit is desired.

46	If a carpenter needs to determine the length of a board to the nearest quarter $(\frac{1}{4})$ inch, he will select as his unit the ______ inch.	quarter
47	Technically, the length of the board will then be stated in terms of a ______ ______ of quarter-inch units.	whole number
48	In practice he will probably state the length of the board in terms of feet, inches and quarter-inches. In measuring and cutting his board, he must know that his unit is the ______ inch.	quarter
49	In a similar way, the machinist specifies a tolerance that must be met. This allowable tolerance is the ______ of measurement that he must recognize and use.	unit

Even the cook is faced with the problem of using units of measurements. When a cake recipe calls for one cup of flour, the flour must be measured to within some specified limits or the cake does not come out right. Maybe the cake making would be more successful if the unit of measurement were specified more carefully.

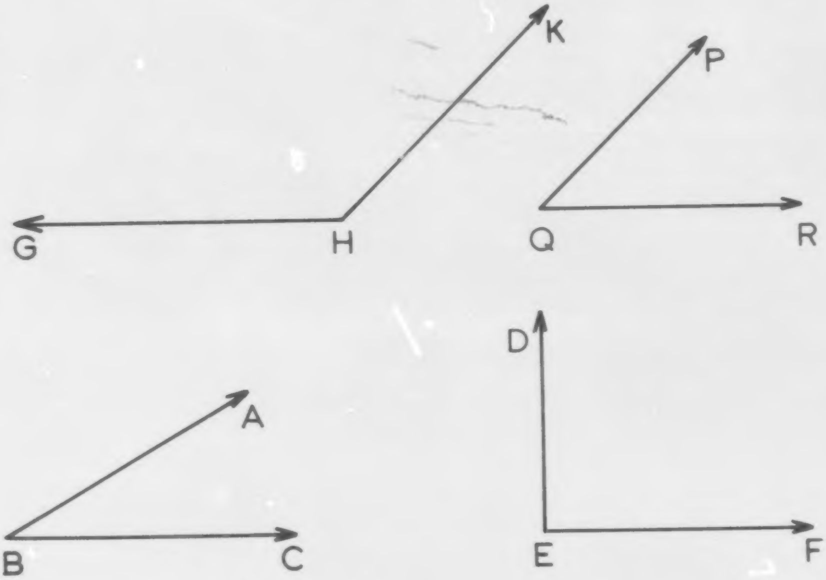
16-8. Measurement of Angles

This sub-program will be brief and what is done will depend on a firm grasp of the idea of measure of a line segment, as discussed in the previous sections, and on the definition of an angle discussed in Chapter 14.

50 Basically, measurement consists of three things.
The first is to select a ______ which will be unit
assigned a value of ______. one or 1

51 The unit used to measure an angle is an angle.
Once the unit is selected it is applied to the
angle, separating it into a number of smaller
parts, each of which is ______ to the unit angle. congruent

52 We finally count the ______ ______ ______ into number of units
which the angle is separated. This count is the
measure of the angle.



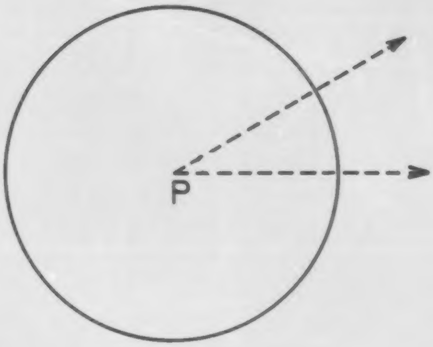
In the picture above, we see several angles any one of which may be used as a unit, or can be measured once a unit is selected. Let the angle ABC be the unit angle.

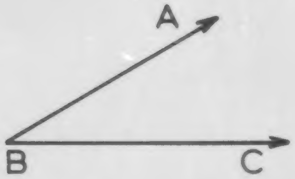
53	Consider the angle DEF. Using $\angle ABC$ as the unit angle, the angle DEF is separated into ______ parts each congruent to the unit angle.	3
54	Hence, the measure of angle DEF is ______.	3
55	Now consider the angle GHK. It is not separated into a whole number of ______ parts by the use of the unit angle ABC.	congruent
56	The measure of $\angle GHK$ is more than ______ but less than ______.	4 5
57	Since the measure of angle GHK is nearer 5 than 4 unit angles, we say that its measure is ______.	5
	Measurement is at best an approximation. The measure of an angle can be determined only to within half a unit. For both line segments and angles the errors can accumulate giving different values for the sum of the measures and the measure of the sum.	

As a convenience in measuring angles a kind of ruler is needed. For example, the face of the clock may serve as a device for measuring angles. Such a device is not called a ruler, but is named a protractor.

The most familiar standard unit of angle measure is the degree. The symbol $^\circ$ is used for the degree. This unit has been used longer and more consistently than any other unit. Other angular units used today are the radian and the mil measure.

By definition, there are 360 degrees in a circle. Consider the circle center P below.





58	If the angle ABC is selected as the unit angle, then it separates the circle into 12 ______ parts.	congruent
59	Consequently, the unit angle contains ______ degrees.	30
60	The measure of an angle is accurate only to within the nearest ______ unit.	half

In summary, we observe that the measurement of an angle is very similar to the measurement of a line segment. In both cases, a unit is selected and this unit is applied to the line segment or angle to be measured. The measure is the number of times that the unit is used.

CHAPTER 17

FACTORS AND PRIMES

This chapter is concerned with whole numbers and some of the properties of whole numbers useful in the study of fractions or rational numbers. While it is not an exercise in computation, it will involve a knowledge of techniques of computation previously presented. Throughout this chapter, "number" will mean "whole number," that is, a member of $\{0, 1, 2, 3, 4, 5, \ldots\}$.

17-1. Products and Factors

1	The product 4×3 is ______.	12
2	The product of 4 and 7 is ______.	28
3	In the mathematical sentence, "$5 \times 7 = 35$," the number 35 is the ______ of 5 and 7.	product
4	If 21 is considered as a product of two whole numbers, the two numbers are ______ and ______.	3; 7
5	7 is a factor of 42 because $6 \times 7 =$ ______.	42
6	6 also is a ______ of 42.	factor
7	Since $5 \times 4 = 20$, either the number ______ or the number ______ is a factor of 20.	5 4
8	Since $3 \times 2 = 6$, then both 3 and 2 are ______ of 6.	factors
9	One also can say that 3 is a factor of 6 because $6 \div 3$ is the whole number ______.	2

10 Factors are involved in the following operations: (Check one or more.)

☐(a) addition ☐(c) multiplication

☐(b) division ☐(d) subtraction

10(a) Incorrect. Factors are used in the operations of multiplication and division.

10(b) Correct.

10(c) Correct.

10(d) Incorrect. See 10(a).

11 8 is a factor of 24 since 24 ÷ 8 = ______. 3

12 1 is a ______ of n since n ÷ 1 = n. factor

13 n is a ______ of n since n ÷ n = 1. factor

14 Definition: a is a factor of b provided
there is a whole number n such that
n × a = ______. b

15 The set of factors of 28 is ______. {1,2,4,7,14,28}

16 The set of factors of 8 is: (Check one.)

☐(a) $\{\frac{1}{2}, 16, 8, 2, 4, 1\}$ ☐(c) {4, 2, 8, 1}

☐(b) {3, 5, 6, 2} ☐(d) {2, 4}

16(a) Incorrect. $\frac{1}{2}$ is not a whole number. Furthermore, 16 is not a factor of 8 since 16 is greater than 8.

16(b) Incorrect. Of these elements only 2 divides 8.

16(c) Correct. Note why the other answers are incorrect.

16(d) Incorrect. 2 and 4 are factors of 8. Since 8 and 1 also are factors of 8, then {2, 4} is not the set of factors of 8.

17 Select a set A each member of which has 6 as a factor:

☐(a) A = {6, 12, 21}

☐(b) A = {12, 36, 72}

☐(c) A = {1, 2, 3, 6}

17(a) There is no whole number n such that $6 \times n = 21$. This response is incorrect.

17(b) Correct. $6 \times 2 = 12$, $6 \times 6 = 36$, $6 \times 12 = 72$. Therefore, 6 is a factor of each member of {12, 36, 72}.

17(c) Since there is no whole number n such that $n \times 6 = 1$, $n \times 6 = 2$, or $6 \times n = 3$, then 6 is not a factor of 1, 2, 3. Hence, this is not a correct response.

Product has been used before as another word for the answer when numbers are multiplied. Factor involves the inverse idea; when two whole numbers are multiplied to obtain a product, each of the numbers used in the multiplication is called a factor of the number which is the product.

We say that 2 is a factor of 14, because we are able to find another whole number, namely 7, which multiplied by 2 gives 14, that is $7 \times 2 = 14$. Given a number such as 36, one can often, by inspection, write all of its factors, including 1 and 36. The factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36.

17-2. Prime Numbers

18	The factors of 7 are ______ and ______.	7; 1
19	The set of factors of 13 is ______.	{13, 1}
20	The set of factors of 7 other than 7 and 1 is ______.	{ }

21	The set of whole numbers less than 10 each member of which has exactly two different factors is ______. The members of this set are called the prime numbers less than 10.	2, 3, 5, 7

Definition: Any whole number which has exactly two different factors is called a prime number. The only factors of a prime number are the number itself and 1.

22	The number one ______ (is, is not) a prime number.	is not
23	The number 1 is not a prime because it does not have exactly two different ______.	factors
24	The next prime number greater than 10 is ______.	11
25	The first two prime numbers greater than 18 are ______ and ______.	19; 23
26	3 × 0 = 0 and 4 × ____ = 0.	0
27	3, 4 and 0 are all ______ of zero.	factors
28	0 is not a prime number because it does not have exactly ______ different factors.	two
29	The set of factors of 6 is ______.	{1, 2, 3, 6}
30	The number 6 is not a ______ number because it has more than two different factors.	prime

31 Which of the following sets have only prime numbers as members? (Check the correct responses.)

☐ (a) {0, 2, 11, 7} ☐ (c) {3, 5, 7, 9}

☐ (b) {1, 3, 5, 7} ☐ (d) {2, 3, 7, 5}

31(a) Incorrect. Zero is not prime.

31(b) Incorrect. One is not a prime since it does not have two different factors.

31(c) Incorrect. 1, 3 and 9 are all factors of 9. Hence, 9 is not prime.

31(d) This is the correct response.

Definition: Any whole number, other than 0 and 1, which is not a prime number is called a composite number. A composite number has at least one factor in addition to itself and 1.

32	The number 15 has the factors 15, 1, 3, 5 and thus is a ______ number.	composite
33	The set of composite numbers less than 12 is ______.	{4,6,8,9,10}
34	All even numbers greater than 2 are ______ numbers, because each has more than two different factors.	composite
35	All numbers in decimal notation which end in 5 or 0, other than 5 and 0, are ______ numbers.	composite
36	Since 3 is a ______ of 6, we say that 6 is a multiple of 3.	factor
37	3 is a factor of 3 and as a consequence, 3 is a multiple of ______.	3

38 Every whole number is a factor of 0, and 0 is a _______ of every whole number.

multiple

39 Every whole number is a multiple of 1, and 1 is a _______ of every whole number.

factor

40 If A is a set of composite numbers, then A could equal:

☐(a) $\{0, 9, 10\}$ ☐(c) $\{4, 6, 15\}$

☐(b) $\{2, 8, 12\}$ ☐(d) $\{6, 9, 21\}$

39(a) Incorrect. 0 is excluded from the list of composite numbers by definition.

39(b) Incorrect. Each member of the set is a multiple of 2, but 2 is a prime number.

39(c) Correct.

39(d) Correct.

In this sub-program the notion of prime number has been introduced as a whole number which has exactly two different factors, the number itself and 1. For example, since $2 = 2 \times 1$, $3 = 3 \times 1$, $13 = 13 \times 1$, then 2, 3, 13 are prime numbers. The number 0 is excluded from the set of primes because it has many factors, that is, $0 = 1 \times 0$, $0 = 2 \times 0$, $\ldots$, $0 = n \times 0$. The number 1 also is excluded since it does not have two distinct whole number factors.

All whole numbers, other than 0, 1 and the prime numbers, are called composite numbers. To be composite, a number must have at least one factor other than 1 and itself. Some composite numbers are easy to recognize, as for example, all multiples of 2 greater than 2.

17-3. Factoring and Prime Factorization

It is frequently desirable to factor a number into more than two factors. For example, $30 = 6 \times 5$, but 6 may be factored as (2×3). Hence, we may write: $30 = 6 \times 5 = (2 \times 3) \times 5 = 2 \times 3 \times 5$ where each of the factors $2, 3, 5$ is a prime.

41	The number 42 may be expressed as 6×7. Express 42 as a product of primes. $42 = \underline{2 \times (\quad \times \quad)}$.	$2 \times (3 \times 7)$
42	If 42 is expressed as $42 = 3 \times 14$, then as a product of primes $42 = \underline{3 \times (\quad \times \quad)}$.	$3 \times (2 \times 7)$
43	Note that the prime factors of 42 are always the same except for the ______ in which they are written. The writing of a number as a product of primes is called the prime factorization of that number.	order
44	Each of the final factors found in Frames 41, 42 and 43 are prime numbers or ______ factors.	prime
45	The prime factorization of 30 is ______.	$2 \times 3 \times 5$ (in any order)
46	The order ______ (is, is not) important, but it is often desirable to write the prime factors in increasing order.	is not
47	$5 \times 11 \times 3 \times 7$ is the prime ______ of 1155.	factorization
48	The number 90 may be factored and written as $90 = 30 \times 3$ or $90 = 2 \times 45$. In each case the prime factorization of 90 is ______.	$2 \times 3 \times 3 \times 5$
49	Some numbers may be expressed as a product of composite numbers. For example, $90 = 6 \times 15$. Factoring these composite factors gives the prime factorization ______.	$2 \times 3 \times 3 \times 5$
50	Each prime factorization of 90 is the ______ and is independent of how it is obtained except for the order in which the prime factors appear in the product.	same

51	$2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5$ is the prime ______ of 1800.	factorization
52	Every composite number can be factored as a product of ______ in exactly one way except for the order in which the factors appear in the product.	primes

We have developed an idea which is fundamental in the study of numbers. That is, any composite number can be factored completely in only one way. Thus, if the prime factors of any number are found, the result will be the same factors, except possibly for order. This has been stated formally in Frame 52, and allows us to speak of *the* prime factorization of a number.

The statement in Frame 52 is called the *Unique Factorization Theorem* and *Fundamental Theorem of Arithmetic*. Only a composite number has a prime factorization; a prime number does not have a prime factorization.

Now let us consider a way of finding the prime factorization of large composite numbers.

53	To test if a prime number such as 3 is a factor of a given number such as 312, *divide* the number by 3, that is, $312 \div 3 =$ ______, a whole number.	104
54	Since $312 =$ ___ × ___, it follows that 3 is a factor of 312.	3 × 104
55	Since 104 is an even number, it has the prime factor 2, that is ___ × ___ $= 104$.	2 × 52
56	$52 \div 2 =$ ______.	26
57	$52 =$ ___ × ___.	2 × 26
58	$26 =$ ___ × ___.	2 × 13
59	Hence, in writing the prime factorization of 104, we must use the prime factor 2 ______ (how many) times.	three

60 The prime factorization of 312 is ______.
In practice it may be convenient to start testing with the smallest prime numbers.

3×2×2×2×13

61 The prime factorization of 714 is: (Select the correct responses.)

☐(a) 6 × 7 × 17
☐(b) 3 × 17 × 2 × 7
☐(c) 34 × 3 × 7
☐(d) 2 × 3 × 7 × 17
☐(e) 1 × 2 × 3 × 7 × 17

61(a) Incorrect. The number 6 is composite.

61(b) Correct. 61(d) also is correct.

61(c) Incorrect. 34 = 2 × 17.

61(d) Correct. 61(b) also is correct.

61(e) Incorrect since the number 1 is not a prime.

62 The prime factorization of 1485 is: (Select the correct responses.)

☐(a) 9 × 3 × 5 × 11
☐(b) 5 × 3 × 9 × 11
☐(c) 3 × 3 × 3 × 11 × 5
☐(d) 5 × 3 × 3 × 11 × 3
☐(e) 1 × 3 × 3 × 3 × 5 × 11

62(a) Incorrect. 9 = 3 × 3.

62(b) Incorrect. The number 9 is composite.

62(c) Correct. 62(d) also is correct.

62(d) Correct. 62(c) also is correct.

62(e) Incorrect since the number 1 is not a prime.

In this sub-program we observed that a composite number may be factored into primes in only one way. Different approaches to finding the factors may give different orders of the factors, but these are not considered different factorizations.

17-4. The Greatest Common Factor of Two Numbers

63	The set of all factors of 72 is ______.	{1,2,4,8,3,6, 12,24,18,36,72}
64	The set of all factors of 30 is ______.	{1,2,3,5,6,10, 15,30}
65	The set of factors common to both 72 and 30 is ______.	{1,2,3,6}
66	The largest single factor which appears in both 72 and 30 is ______. We call 6 the greatest common factor (abbreviated g.c.f.) of 72 and 30.	6
67	The prime factorizations of 30 and 72 are $30 = 2 \times 3 \times 5$ $72 = 2 \times 2 \times 2 \times 3 \times 3$ Let us pair factors from their prime factorizations, thus: $30 = (2 \times 3) \times 5$ $72 = 2 \times 2 \times (2 \times 3) \times 3$ The greatest common factor is ____ × ____ = ____.	$2 \times 3 = 6$
68	The prime factorizations of 72 and 54 are: $72 = 2 \times 2 \times 2 \times 3 \times 3 = (2 \times 3 \times 3) \times 2 \times 2$ $54 = 2 \times 3 \times 3 \times 3 = (2 \times 3 \times 3) \times 3$ The g.c.f. is found to be ____ × ____ × ____ = ____.	$2 \times 3 \times 3 = 18$
69	The set of all factors of 8 is ______, and the set of all factors of 9 is ______.	{1,2,4,8} {1,3,9}
70	The g.c.f. of 8 and 9 is ______.	1

71	The prime factorization of 8 is ______ and the prime factorization of 9 is ______. Note that the greatest common factor of 8 and 9 cannot be obtained by taking pairs of factors from the prime factorization.	2 × 2 × 2 3 × 3
72	The set of all factors of 7 is ______.	{1,7}
73	The set of all factors of 49 is ______.	{1,7,49}
74	The g.c.f. of 7 and 49 is ______.	7
75	The prime factorization of 49 is ______.	7 × 7
76	Since 7 is a ______ number, it does not have a prime factorization. Hence, the g.c.f. of 7 and 49 cannot be obtained by taking pairs of factors from the prime factorization.	prime
77	The set of all factors of 5 is ______.	{1,5}
78	The set of all factors of 7 is ______.	{1,7}
79	The g.c.f. of 5 and 7 is ______.	1
80	Neither 5 nor 7 have prime factorizations since they both are ______.	prime
81	Frames 71 - 80 indicate the three instances in which the greatest common factor of two numbers cannot be obtained by taking pairs of factors from the prime ______.	factorization
82	These cases occur when the greatest common factor is 1, or when at least one number is a prime. The g.c.f. of 5 and 25 is ______. The g.c.f. of 5 and 25 cannot be obtained from the prime factorization because 5 is a prime.	5

83	The g.c.f. of 10 and 21 is ______.	1
84	Although 10 and 21 are composite numbers, the g.c.f. could not be obtained from the prime ______.	factorization
85	Since 7 and 11 are primes, the g.c.f. cannot be obtained from the ______ ______.	prime factorization
86	It should be noted, however, that when the prime factorization of two numbers does not yield the g.c.f., then the g.c.f. is either ______ or one of the two numbers.	1

The greatest common factor may be used in reducing a fraction to the lowest form.

17-5. The Least Common Multiple of Two Numbers

The notion of the least common multiple of two or more numbers appears as the lowest common denominator in addition and subtraction of fractions.

87	If one considers a number such as 5 and multiplies it successively by the members of the set of whole numbers, the result of this multiplication is the set ______.	{0,5,10,15,20 25, ...}
88	The set of multiples of 3 is ______.	{0,3,6,9,12, 15, ...}
89	The set of multiples of 4 is ______.	{0,4,8,12,16, 20, ...}
90	12, 24 and 36 are common ______ of 3 and 4.	multiples

91 The smallest common multiple of 3 and 4 (other than zero) of the sets of multiples of 3 and 4 is ______. The number 12 is called the least common multiple (abbreviated l.c.m.) of 3 and 4.

12

92 The set of multiples of 15 is

{0, 15, 30, 45, 60, 75, 90, ...} .

The set of multiples of 6 is

{0, 6, 12, 18, 24, 30, 36, ...} .

The least common multiple of 15 and 6 is ______.

30

93 The set of multiples of 2 is

{0, 2, 4, 6, 8, 10, ...} ,

and the set of multiples of 3 is

{0, 3, 6, 9, 12, ...} .

6 is the ______ ______ ______ of 2 and 3.

least common multiple

94 The least common multiple of 6 and 10 is:

☐(a) 60 ☐(c). 0

☐(b) 30 ☐(d) 20

94(a) Incorrect. While 60 is a common multiple of both numbers, it is not the l.c.m.

94(b) Correct. This is the least number (other than zero) which is common to both sets of multiples of 6 and 10.

94(c) Incorrect. 0 is a common multiple of both 6 and 10, but the definition requires a common multiple to be greater than zero.

94(d) Incorrect since 20 is not a multiple of 6.

95 Writing the sets of multiples to obtain the l.c.m. is sometimes less economical than employing prime factorization. The prime factorizations of 15 and 6 are:

15 = 3 × 5
6 = 2 × 3

Any multiple of 15 must contain the factors 3 and 5 and any multiple of 6 must contain the factors 2 and 3. Hence, any common multiple of 15 and 6 must contain the factors ______. 3, 5, 2

96 Some possible products containing the factors of 6 and 15 are:

3 × 5 × 2 × 2 = 60
3 × 5 × 2 × 3 = 90
3 × 5 × 2 = 30
3 × 5 × 2 × 0 = 0
3 × 5 × 2 × 7 = 210.

All of these are ______ multiples of 6 and 15. common

97 The least common multiple of 6 and 15 is ______. 30

98 The prime factorization of 198 is 2 × 3 × 3 × 11 and the prime factorization of 42 is 2 × 3 × 7. Find the least common multiple of 198 and 42 by writing the prime factorization of 198 and multiply it by the part of the prime factorization of 42 not included in that of 198. The prime factorization of the least common multiple of 198 and 42 is: (Check all correct responses.)

☐(a) 2 × 3 × 3 × 11 × 2 × 3 × 7
☐(b) 2 × 3 × 11 × 7
☐(c) 2 × 3 × 3 × 11 × 7
☐(d) 2 × 3 × 7 × 3 × 11

98(a) Incorrect. The prime factorization of 198 includes the 2 × 3 from the prime factorization of 42. The only factor of the prime factorization of 42 not included in the prime factorization of 198 is 7.

98(b) Incorrect. The factor 3 must appear twice since it appears twice in one of the prime factorizations of one of the numbers.

98(c) Correct. See 98(d).

98(d) Correct, but in this ordering of factors, the l.c.m. is more difficult to recognize.

To find the least common multiple of two numbers, first find the prime factorization of one of the numbers, then multiply it by the part of the prime factorization of the other number not included in that of the first.

99	Since both 7 and 17 are prime numbers, the l.c.m. of 7 and 17 is ______.	119
100	The l.c.m. of 12 and 12 is ______.	12
101	The prime factorization of the l.c.m. of 7 and 42 is ______, since 7 is prime and the prime factorization of 42 is 2 × 3 × 7.	7 × 3 × 2
102	The prime factorization of the l.c.m. of 7 and 62 is ______.	7 × 2 × 31

If one or both of two numbers are primes, these are used as factors in the prime factorization of the l.c.m.

Multiples of numbers can be found by multiplying the number by 0, by 1, by 2, by 3, et cetera. Multiples of 8 are 0, 8, 16, 24, 32, 40, 48, ... and multiples of 9 are 0, 9, 18, 27, 36, 45, ... The least common multiple can be found by this method, but may require writing down many terms of each sequence before a common one is found.

Because of the tediousness of writing these terms, the use of prime factorization for finding the l.c.m. is more convenient.

17-6. *Chapter Summary*

The notions developed in this chapter have been based on *factors* of whole numbers. We have considered finding the factors of a given number and also of finding a number if we know the factors. A factor of a number implies a multiplication and we say that a is a factor of b provided there is a whole number n such that $n \times a = b$. We also may determine whether or not a is a factor of b by dividing b by a. If the result is a whole number n, then we know that $n \times a = b$ and a is a factor of b.

A prime number is defined as any whole number which has *exactly two different* factors. This excludes 0 and 1 and all numbers which have more than two factors. This enables us to write the set of primes as follows:

$$\{2, 3, 5, 7, 11, 13, 17, 19, \ldots\} .$$

All other whole numbers, except 0 and 1, form the set of *composite* numbers which may be written as:

$$\{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, \ldots\} .$$

For a consideration of the various factors of composite numbers, one arrives at a conclusion called *The Fundamental Theorem of Arithemtic*. This theorem is stated as follows:

> Every composite number can be factored as a product of primes in exactly one way except for the order in which the prime factors appear in the product.

This statement also is known as the *Unique Factorization Theorem*.

Thus, we are able to consider any composite number and write its prime factorization. For example,

$$90 = 2 \times 3 \times 3 \times 5 \quad \text{and} \quad 1092 = 2 \times 2 \times 3 \times 7 \times 13 .$$

The fact that each composite number has a unique prime factorization is useful in a number of places in arithmetic and algebra. In this chapter we used this fact to find the greatest common factor of two numbers and to find the least common multiple of two or more numbers. The greatest common factor is found useful in the simplification of fractions. The least common multiple of two or more numbers is used in addition and subtraction of fractions.

CHAPTER 18

INTRODUCING RATIONAL NUMBERS

All our work with numbers up to this point has been with the set of whole numbers. We have pretended as if they are the only numbers which exist. We have considered the operations of addition and multiplication in the set of whole numbers and have studied properties of these operations.

18-1. Introducing Rational Numbers

1	5 + 2 = ______ .	7
2	5 + 2 represents a member of the set of whole ______.	numbers
3	The sum of two whole numbers is always a ______ number.	whole
4	Since the sum of two whole numbers is a whole number, the set of whole numbers is ______ under the operation of addition.	closed
5	5 × 2 = ______.	10
6	5 × 2 represents a member of the set of ______ numbers.	whole
7	The product of two whole numbers is always a ______ ______.	whole number
8	Since the product of two whole numbers is a whole number, the set of whole numbers is ______ under the operation of multiplication.	closed

Recall from Chapter 9 a definition of the operation division. Let a, b and n represent whole numbers, where b is not zero. Division may be defined in terms of multiplication as follows:

Definition: $a \div b = n$ if and only if $a = b \times n$.

9	For example $8 \div 2 = 4$ since $8 = 2 \times$ ______.	4
10	$12 \div 4 =$ ______ since $12 = 4 \times 3$.	3
11	$144 \div 6 = 24$ since $144 = 6 \times$ ______.	24
12	$17 \div 6 =$ ______. (whole number)	Impossible. There is no whole number n such that $17 = 6 \times n$.
13	The quotient of two whole numbers ______ (is, is not) always a whole number.	is not
14	The set of whole numbers is not ______ under the operation of division.	closed

15 Eight pieces of candy are to be divided equally among four boys. Each boy will receive:

☐ (a) 3 pieces
☐ (b) 2 pieces
☐ (c) 1 piece
☐ (d) cannot be done

15(a) This response is incorrect since $4 \times 3 = 12$, not 8.

15(b) This response is correct since $4 \times 2 = 8$.

15(c) This response is incorrect since $4 \times 1 = 4$, not 8.

15(d) $4 \times 2 = 8$ and therefore 14(b) is the correct response.

16 Seven pieces of candy are to be divided equally among three girls. Each girl will receive:

☐ (a) 3 pieces ☐ (c) 0 pieces

☐ (b) 2 pieces ☐ (d) cannot be done

16(a) This response is incorrect since $3 \times 3 = 9$, not 7.

16(b) This response is incorrect since $3 \times 2 = 6$, not 7.

16(c) This response is incorrect since $3 \times 0 = 0$, not 7. But if the girls wait until this problem is solved in the set of whole numbers, this response might be correct.

16(d) This response is correct since there is no whole number n such that $3 \times n = 7$.

17 If $5 \div 2 = n$, then n is a member of the set of: (One of the two responses is correct.)

☐ (a) whole numbers

☐ (b) rational numbers

17(a) This response is incorrect since $2 \times n \neq 5$ if n is a whole number.

17(b) This response is correct. $2 \times \frac{5}{2} = 5$ and "five-halves," written $\frac{5}{2}$, is a rational number representing five divided by two.

We have now used whole numbers such as a and b, with b not zero, in the form $\frac{a}{b} = a \div b$. A number in this form is called a fraction, and is one way of indicating a rational number.

18 The rational number representing $7 \div 3$, if written in the form $\frac{a}{b}$, is ______. $\frac{7}{3}$

18

19 The number represented by $\frac{8}{5}$ is a member of the set of:

☐ (a) counting numbers

☐ (b) whole numbers

☐ (c) rational numbers

19(a) Incorrect. The set of counting numbers is denoted by $\{1, 2, 3, 4, \ldots\}$ and $\frac{8}{5}$ does not belong to this set.

19(b) Incorrect. The set of whole numbers is denoted by $\{0, 1, 2, 3, \ldots\}$ and $\frac{8}{5}$ does not belong to this set.

19(c) Correct since $\frac{8}{5}$ represents a rational number and $\frac{8}{5} = 8 \div 5$.

20 If n represents a whole number and $n \neq 0$, then $0 \div n = 0$ since $n \times 0 =$ _____. 0

21 The number sentence $0 \div b = 0$ is true if b is any element of the following set: (Check all correct responses.)

☐ (a) {3, 6, 9} ☐ (b) {0, 2, 4} ☐ (c) {1, 2, 3}

21(a) Correct since b can be any whole number except 0. 21(c) also is correct.

21(b) Incorrect since b cannot be 0. Recall that division by 0 is undefined.

21(c) Correct since b can be any whole number except 0. 21(a) also is correct.

22 Which of the following is a set of rational numbers:

☐ (a) $\{\frac{5}{2}, \frac{3}{0}, \frac{4}{6}\}$ ☐ (b) $\{\frac{2}{2}, \frac{7}{3}, \frac{0}{5}\}$

22(a) Incorrect. $\frac{3}{0}$ does not represent a rational number since $3 \div 0$ is undefined.

22(b) Correct. Each member of the set is a number of the form $\frac{a}{b}$, where a and b are whole numbers and $b \neq 0$.

23 If a belongs to A, and b belongs to B, then $\frac{a}{b}$ represents a rational number when:

☐ (a) A = {1, 8} and B = {4, 3}

☐ (b) A = {0, 6} and B = {2, 30}

☐ (c) A = {5, 7} and B = {0, 4}

23(a) Correct. All the fractions $\frac{1}{4}$, $\frac{1}{3}$, $\frac{8}{4}$ and $\frac{8}{3}$ represent rational numbers. 23(b) also is correct.

23(b) Correct. All the fractions $\frac{0}{2}$, $\frac{0}{30}$, $\frac{6}{2}$ and $\frac{6}{30}$ represent rational numbers. 23(a) also is correct.

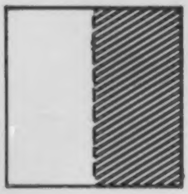
23(c) Incorrect. $\frac{5}{0}$ and $\frac{7}{0}$ do not represent rational numbers, since b cannot be 0 according to our definition of division.

18-2. Models for Rational Numbers

In setting up physical models for rational numbers we usually begin by fixing some "basic unit," for example, a segment, a rectangular region, a circular region, or a collection of identical things. This unit is then separated into a certain number of "congruent" parts. These parts, compared to the unit, give us the basis for a model for rational numbers.

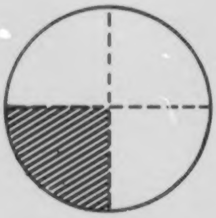
24 In the model below, if the "basic unit" is the square region, then the part shaded represents one of the ________ (how many) congruent parts?

2 or two



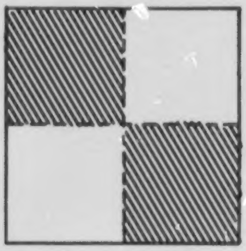
25 In the model below, the part shaded represents one of the ______ congruent parts of the basic unit, the circular region.

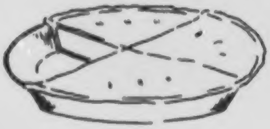
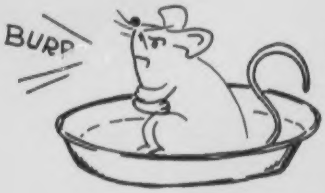
4 or four



26 In the model below, the part shaded represents ______ of the four congruent parts.

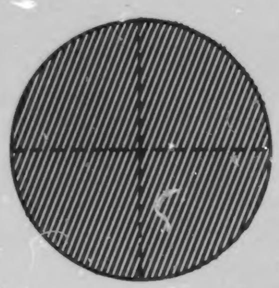
2 or two



27	The rational number $\frac{a}{b}$ is used to represent "a" of the "b" congruent parts of some basic unit. Thus, $\frac{2}{5}$ represents 2 of the ______ congruent parts.	5
28	The rational number $\frac{5}{2}$ represents 5 of the ______ congruent parts. (It is apparent that the basic unit is used several times.)	2
29	The rational number $\frac{7}{7}$ represents ______ of the 7 congruent parts.	7
30	The rational number $\frac{0}{3}$ represents ______ of the 3 congruent parts.	0
31	Does the rational number $\frac{0}{4}$ represent nothing? ______ (yes, no)	No. $\frac{0}{4}$ represents none of 4 congruent parts.
32	The part of a pie in the pan is $\frac{\_\_\_}{4}$ of a pie. 	$\frac{3}{4}$
33	The model below represents $\frac{0}{4}$ of a pie. The $\frac{0}{4}$ ______ mean there is nothing in the pan. (does, does not)  	does not

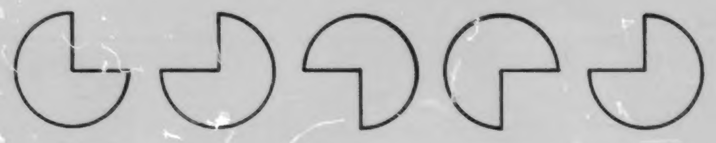
34 In the model below, the shaded region can be represented by the rational number _______.

$\frac{4}{4}$

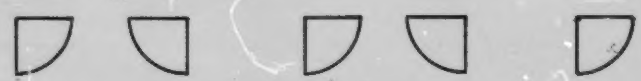


35 Which of the following is a model for $\frac{5}{4}$ of a circular region:

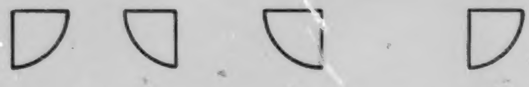
☐ (a)



☐ (b)



☐ (c)



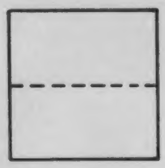
35(a) Incorrect. There are 5 congruent regions, but each region is not $\frac{1}{4}$ of a circular region.

35(b) Correct. There are 5 congruent regions, and each is $\frac{1}{4}$ of a circular region. Note why 35(a) and 35(c) are incorrect.

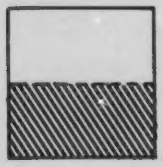
35(c) Incorrect. Each region is $\frac{1}{4}$ of a circular region, but there are only 4 of them, not 5.

The square region to the right represents the "basic unit" in Frames 36 - 40.

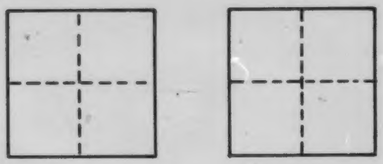
36 Shade a region (or regions) which would represent the rational number $\frac{1}{2}$.



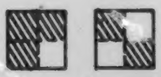
Shade any 1 of the rectangular regions, but not both. For example:



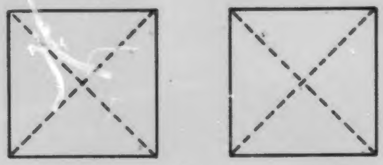
37 Shade a region (or regions) which would represent the rational number $\frac{5}{4}$.



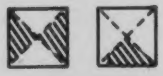
Shade any 5 of the smaller square regions. For example:



38 Shade a region (or regions) which would represent the rational number $\frac{3}{4}$.

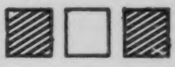


Shade any 3 of the triangular regions. For example:

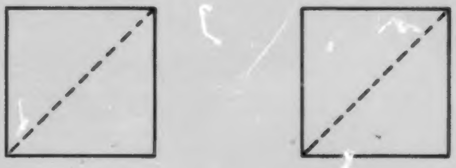


39 Shade a region (or regions) which would represent the rational number $\frac{2}{3}$.

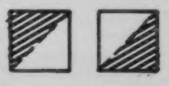
Shade any 2 of the square regions. For example:



40 Shade a region (or regions) which would represent the rational number $\frac{2}{2}$ so that $\frac{2}{2} = 1$.



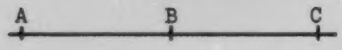
Shade any 2 of the triangular regions. For example:



The numbers for which our regions are models are called rational numbers. The particular numeral form in which they are often expressed is called a fraction. In general the "fractional form" $\frac{a}{b}$ represents a "rational number" provided a is a whole number and b is some whole number other than zero, that is a counting number.

Referring to our models, we see that b, the denominator, always designates the number of congruent parts into which the basic unit has been partitioned; while a, the numerator, indicates how many of these congruent parts are to be considered.

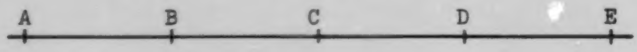
41 Consider the segment $\overline{AC}$ below.



The segment $\overline{AB}$ is congruent to the segment $\overline{BC}$, written as $\overline{AB} \cong \overline{BC}$. Hence, the measure of segment $\overline{AB}$ is the same as the measure of segment $\overline{BC}$. And the measure of segment $\overline{AB}$ is $\frac{1}{2}$ the measure of segment ______.

$\overline{AC}$

42 Consider the segment $\overline{AE}$ below.



If $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE}$, then the measure of $\overline{AE}$ = ______ the measure of $\overline{AD}$.

$\frac{4}{3}$

43 $m(\overline{AE})$ = _____ $m(\overline{BD})$.

$\frac{4}{2}$

44 $m(\overline{DA})$ = _____ $m(\overline{AE})$.

$\frac{3}{4}$

In Figure 18.1 below, the segment whose endpoints A and B are labeled zero and one has been partitioned into twelve congruent segments.

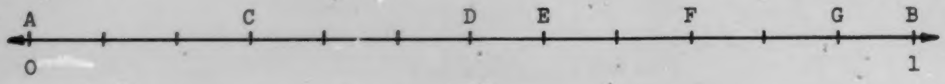


Figure 18.1

45 The rational number $\frac{3}{12}$ is associated with the point ______.

C

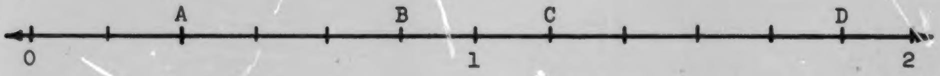
46 The rational number $\frac{7}{12}$ is associated with the point ______. E

47 The rational number $\frac{9}{12}$ is associated with the point ______. F

48 The rational number $\frac{12}{12}$ may be associated with the point ______. B

49 The rational number $\frac{0}{12}$ may be associated with the point ______. A

In the model below, label the indicated points A, B, C and D with appropriate rational numbers for Frames 50 - 53.



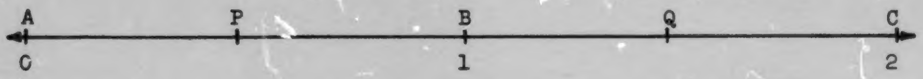
50 Point A is labeled with the rational number ______. $\frac{2}{6}$

51 Point B is labeled with the rational number ______. $\frac{5}{6}$

52 Point C is labeled with the rational number ______. $\frac{7}{6}$

53 Point D is labeled with the rational number ______. $\frac{11}{6}$

Consider the following model of a number line for Frames 54 - 63.



54	If $\overline{AP} \cong \overline{PB}$, then point ______ may be labeled with the fraction $\frac{1}{2}$.	P
55	If segment $\overline{AB}$ is partitioned into 4 congruent segments, the fraction $\frac{\quad}{4}$ may be used to label the point P.	$\frac{2}{4}$
56	If segment $\overline{AB}$ is partitioned into 6 congruent segments, the fraction $\frac{\quad}{6}$ may be used to label the point P.	$\frac{3}{6}$
57	If segment $\overline{AB}$ is partitioned into 8 congruent segments, the fraction $\frac{\quad}{8}$ may be used to label the point P.	$\frac{4}{8}$
58	Write the set of all fractions any one of which may be used to label the point P: $\{\frac{1}{2}, \frac{\quad}{4}, \frac{\quad}{6}, \frac{\quad}{8}, \frac{\quad}{10}, \frac{\quad}{12}, \ldots\}$	$\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \ldots\}$
59	If point Q is such that $\overline{BA} \cong \overline{QC} \cong \overline{AP}$, then point Q may be labeled with $\frac{\quad}{2}$.	$\frac{3}{2}$
60	If segments $\overline{AB}$ and $\overline{BC}$ are partitioned into 4 congruent segments each, the fraction $\frac{\quad}{4}$ may be used to label the point Q.	$\frac{6}{4}$
61	If segments $\overline{AB}$ and $\overline{BC}$ are partitioned into 6 congruent segments each, the fraction $\frac{\quad}{6}$ may be used to label the point Q.	$\frac{9}{6}$

62 If segments $\overline{AB}$ and $\overline{BC}$ are partitioned into 8 congruent segments each, the fraction _______ may be used to label the point Q.

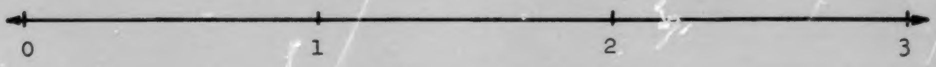
$\frac{12}{8}$

63 Write the set of all fractions, any one of which may be used to label the point Q:

$\{\frac{\ }{2}, \frac{\ }{4}, \frac{\ }{6}, \frac{\ }{8}, \frac{\ }{10}, \frac{\ }{12}, \ldots\}$.

$\{\frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \frac{15}{10}, \frac{18}{12}, \ldots\}$

Consider the following model of a number line for Frames 64 - 74.



64 The point labeled with the whole number 0 may be labeled with the fraction $\frac{0}{1}$ since $0 \div 1 =$ _____.

0

65 The point labeled with the whole number 0 may be labeled with the fraction $\frac{0}{2}$ since $0 \div$ _____ $= 0$.

2

66 The point labeled with the whole number 0 may be labeled with the fraction _______ since $0 \div 3 = 0$.

$\frac{0}{3}$

67 In general, if k is any counting number, the point labeled with the whole number 0 may be labeled with the fraction $\frac{0}{k}$ since _______ $\div$ _______ $= 0$.

$0 \div k = 0$

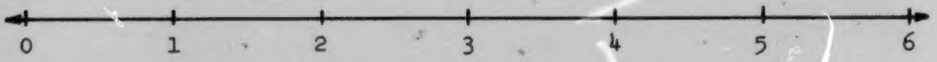
68 Write the set of all fractions, any one of which may be used to label the point labeled with the whole number 0:

{ _______________________ }.

$\{\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4}, \ldots\}$

69	There are ____________________ (a finite number of, infinitely many) fractions in the set in Frame 68 since there is no last counting number k.	infinitely many
70	The point labeled with the whole number 1 may be labeled with the fraction $\frac{1}{1}$ since $1 \div 1 =$ ______.	1
71	The point labeled with the whole number 1 may be labeled with the fraction $\frac{2}{2}$ since $2 \div 2 =$ ______.	1
72	The point labeled with the whole number 1 may be labeled with the fraction ______ since $3 \div 3 = 1$.	$\frac{3}{3}$
73	May the fraction $\frac{4}{4}$ be used to label the point labeled with the whole number 1? ______ (yes, no)	Yes, since $4 \div 4 = 1$
74	In general, if n is any counting number, then the point labeled 1 may be labeled with the fraction $\frac{n}{n}$ since _____ $\div$ _____ $=$ _____.	$n \div n = 1$

Consider the following model of a number line for Frames 75 - 80.



75	The point labeled with the whole number 2 may be labeled with the fraction $\frac{2}{1}$ since _____ $\div$ _____ $=$ _____.	$2 \div 1 = 2$

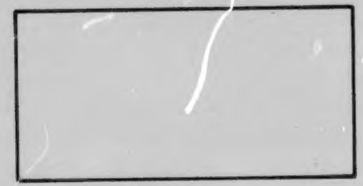
76	Write the set of all fractions, any one of which may be used to label the point labeled with the whole number 2. $\{\frac{2}{1}, \_, \_, \_, \_, \_, \ldots\}$	$\{\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \frac{10}{5}, \frac{12}{6}, \ldots\}$
77	The point labeled with the whole number 3 may be labeled with the fraction $\frac{3}{1}$ since ______ + ______ = ______.	3 + 1 = 3
78	The point labeled with the whole number 4 may be labeled with the fraction ______ since 4 + 1 = 4.	$\frac{4}{1}$
79	The point labeled with the whole number 5 may be labeled with the fraction $\frac{5}{1}$ since ______ + ______ = ______.	5 + 1 = 5
80	In general, if n is any whole number, the point labeled with n may be labeled with the fraction $\frac{n}{1}$ since ______ + ______ = ______.	n + 1 = n

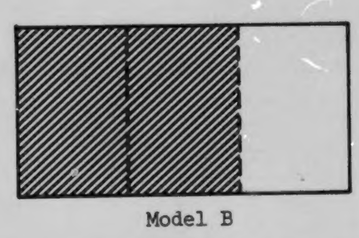
CHAPTER 19

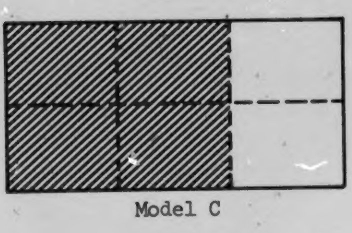
EQUIVALENT FRACTIONS

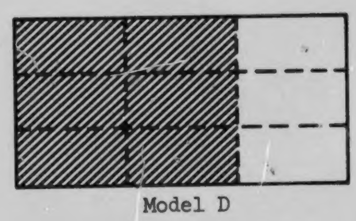
We have developed models for rational numbers from two different points of view, namely, unit regions and the number line. We have noted that fractions of the form $\frac{a}{b}$ name such numbers, with the counting number $\underline{b}$ designating how many congruent parts the unit region or segment is partitioned into and the whole number $\underline{a}$ designating how many of these congruent parts are being considered.

19-1. Equivalent Fractions


Model A


Model B


Model C


Model D

In the following frames, Model A represents the unit region.

1	In Model B the unit region has been partitioned into 3 congruent parts and ______ of these parts are shaded.	2
2	Hence, Model B is a model for the rational number ______.	$\frac{2}{3}$
3	In Model C the unit region has been partitioned into ______ congruent parts and 4 of these parts are shaded.	6

4 Hence, Model C is a model for the rational number ______. $\frac{4}{6}$

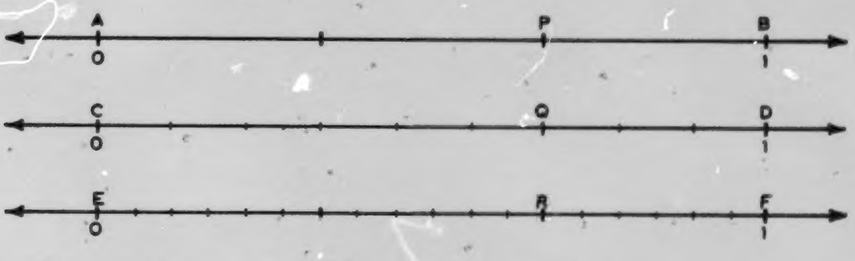
5 In Model D the unit region has been partitioned into ______ congruent parts and ______ of these parts are shaded. 9; 6

6 Hence, Model D is a model for the rational number ______. $\frac{6}{9}$

7 Since the shaded portions in Models B, C and D are congruent and have the same measures, the numbers representing them are the same.

Hence, $\frac{2}{3}$, $\frac{4}{6}$ and $\frac{6}{9}$ are different names for the ______ rational number. same

In the models below, the segments $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ are congruent and each has the measure 1.



$\overline{AB}$ is partitioned into 3 congruent segments;
$\overline{CD}$ is partitioned into 9 congruent segments; and
$\overline{EF}$ is partitioned into 18 congruent segments.

8 Since segment $\overline{AP}$ is 2 of the 3 congruent parts of segment $\overline{AB}$, the measure of $\overline{AP}$ is $\frac{2}{3}$ the measure of $\overline{AB}$ and the fraction $\frac{2}{3}$ is associated with point ______. P

9	Since segment $\overline{CQ}$ is ______ of the 9 congruent parts of segment $\overline{CD}$, the measure of $\overline{CQ}$ is $\frac{6}{9}$ the measure of $\overline{CD}$ and the fraction ______ is associated with point Q.	6 $\frac{6}{9}$
10	Since segment $\overline{ER}$ is ______ of the 18 congruent parts of segment $\overline{EF}$, the measure of $\overline{ER}$ is ______ the measure of $\overline{EF}$ and the fraction $\frac{12}{18}$ is associated with the point ______.	12 $\frac{12}{18}$ R
11	Since $\overline{AP} \cong \overline{CQ} \cong \overline{ER}$, the measures of these segments are the same. Hence, the fractions $\frac{2}{3}$, $\frac{6}{9}$ and $\frac{12}{18}$ are different names for the same ______ number.	rational
12	Each member of $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{12}{18}\}$ is a different name for the ______ rational number.	same
13	If $\frac{2}{3} = \frac{2 \times 2}{3 \times n} = \frac{4}{6}$, then n = ______ since 3 × 2 is another name for 6.	2
14	If $\frac{2}{3} = \frac{2 \times n}{3 \times 3} = \frac{6}{9}$, then n = ______.	3
15	If $\frac{8}{12} = \frac{4 \times 2}{6 \times 2} = \frac{n}{6}$, then n = ______.	4
16	If $\frac{10}{15} = \frac{2 \times n}{3 \times n} = \frac{2}{3}$, then n = ______.	5
17	If $\frac{8}{12} = \frac{4 \times 2}{6 \times 2} = \frac{n}{6}$, then n = ______.	4
18	If $\frac{4}{6} = \frac{2 \times 2}{3 \times 2} = \frac{2}{n}$, then n = ______.	3
19	$\frac{2}{9}$ and $\frac{2 \times 5}{9 \times 5}$ are different names for the same ______ ______.	rational number

19

20 $\frac{0}{5}$ and $\frac{0 \times 7}{5 \times 7}$ are different names for the ______ rational number.

same

21 $\frac{5}{3}$ and $\frac{35}{21}$ are ______ (the same, different) names for the same rational number.

different

22 The sentence $\frac{a}{b} = \frac{a \times k}{b \times k}$ is true provided k belongs to the set of:

☐(a) counting numbers ☐(b) whole numbers

22(a) This response is correct since k can be any member of the set $\{1, 2, 3, 4, \ldots\}$.

22(b) This response is incorrect. Since 0 belongs to the set of whole numbers, $\frac{a \times 0}{b \times 0} = \frac{0}{0}$ and division by 0 is undefined.

23 $\frac{35}{20}$ is another name for the rational number:

☐(a) $\frac{5}{3}$ ☐(b) $\frac{7}{4}$ ☐(c) $\frac{9}{5}$

23(a) This response is incorrect since $5 \times 7 = 35$, but $3 \times 7 \neq 20$.

23(b) This response is correct since $\frac{7}{4} = \frac{7 \times 5}{4 \times 5} = \frac{35}{20}$.

23(c) This response is incorrect since $5 \times 4 = 20$, but $9 \times 4 \neq 35$.

24 The fractional form $\frac{7 \times k}{13 \times k}$ is called a higher form of the fraction ______ where k is a counting number greater than 1.

$\frac{7}{13}$

25 The highest form of the fraction $\frac{2}{3}$ is:

☐(a) $\frac{2}{3}$

☐(c) $\frac{12}{18}$

☐(b) $\frac{4}{6}$

☐(d) non-existent

25(a) Incorrect since the fraction $\frac{4}{6}$ is a higher form.

25(b) Incorrect since the fraction $\frac{12}{18}$ is a higher form.

25(c) Incorrect since the fraction $\frac{20}{30}$ is a higher form.

25(d) This response is correct since $\frac{2}{3} = \frac{2 \times k}{3 \times k}$ and there is no greatest counting number k since $k + 1$ is greater than k for any counting number k.

26 The fractional form $\frac{a \times k}{b \times k}$ where $k > 1$ is called a higher form of the fraction _______.

$\frac{a}{b}$

27 A highest form of the fraction $\frac{a}{b}$ _____________ (does, does not) exist.

does not

28 Two lower forms of the fraction $\frac{75}{50}$ are the fractions _______ and _______.

$\frac{3}{2}$; $\frac{15}{10}$

29 The lowest form of the fraction $\frac{75}{50}$ is the fraction _______.

$\frac{3}{2}$

30 The lowest form of the fraction $\frac{546}{728}$ is:

☐(a) $\frac{273}{364}$ ☐(c) $\frac{39}{52}$

☐(b) $\frac{78}{104}$ ☐(d) none of these

30(a) Incorrect since $\frac{273}{364}$ is a lower form but not the lowest form of $\frac{546}{728}$. See 30(d).

30(b) Incorrect since $\frac{78}{104}$ is a lower form but not the lowest form of $\frac{546}{728}$. See 30(d).

30(c) Incorrect since $\frac{39}{52}$ is a lower form but not the lowest form of $\frac{546}{728}$. See 30(d).

30(d) This response is correct since $\frac{3}{4}$ is the lowest form of the fraction $\frac{546}{728}$.

In this sub-program we have developed the idea that any rational number can be represented by different fractions which are said to be equivalent. Any fraction can be changed to an equivalent fraction "in higher terms" by multiplying both numerator and denominator by the same counting number k where $k > 1$. Since $\frac{a}{b} = \frac{a \times k}{b \times k}$ for any counting number k, there is no highest form of the rational number $\frac{a}{b}$ since there is no greatest counting number k.

19-2. Equivalent Fractions in Lower Terms

31 Since $15 + 1 = 15$, $15 + 3 = 5$, $15 + 5 = 3$ and $15 + 15 = 1$, any member of $\{1, 3, 5, 15\}$ is a factor of the number ______. 15

32	15 + n is a whole number provided n is a ______ of the number 15.	factor
33	The set of all factors of 10 is ______.	{1,2,5,10}
34	The set of numbers common to {1, 3, 5, 15} and {1, 2, 5, 10} is ______.	{1,5}
35	{1, 5} is the set of all ______ factors of 15 and 10.	common
36	Since {1, 5} is the set of common factors of 15 and 10, the greatest common factor of 15 and 10 is ______.	5
37	The greatest common factor of two non-zero whole numbers n and m is the greatest number in the set of all ______ ______ of n and m.	common factors

38 The lowest form of the fraction $\frac{36}{48}$ is:

☐(a) $\frac{18}{24}$ since $\frac{36}{48} = \frac{18 \times 2}{24 \times 2}$

☐(b) $\frac{12}{16}$ since $\frac{36}{48} = \frac{12 \times 3}{16 \times 3}$

☐(c) $\frac{6}{8}$ since $\frac{36}{48} = \frac{6 \times 6}{8 \times 6}$

☐(d) $\frac{3}{4}$ since $\frac{36}{48} = \frac{3 \times 12}{4 \times 12}$

38(a) This response in incorrect. See 38(d).

38(b) This response is incorrect. See 38(d).

38(c) This response is incorrect. See 38(d).

38(d) This response is correct. Note that 2, 3, 6 and 12 are common factors of 36 and 48 and that 12 is the greatest common factor of 36 and 48. Hence $\frac{3}{4}$ is the lowest form of the fraction $\frac{36}{48}$.

39	$\frac{2}{3}$ is called the lowest form of the fraction $\frac{14}{21}$ since 7 is the ______ common factor of 14 and 21.	greatest
40	The set of all common factors of 8 and 20 is ______.	{1,2,4}
41	The greatest number {1, 2, 4} is ______.	4
42	Hence, the lowest form of $\frac{20}{8}$ is ______.	$\frac{5}{2}$
43	The set of all common factors of 18 and 24 is ______.	{1,2,3,6}
44	The greatest common factor of 18 and 24 is ______.	6
45	Hence, the lowest form of $\frac{18}{24}$ is ______.	$\frac{3}{4}$
46	Write the set of all common factors of 28 and 42. {______}	{1,2,7,14}
47	The greatest common factor of 28 and 42 is ______.	14
48	Hence, ______ is the lowest form of $\frac{42}{28}$.	$\frac{3}{2}$

49 $\frac{a}{b}$ is a lower form of $\frac{a \times k}{b \times k}$ if $k \neq 1$ and k belongs to the set of:

☐(a) All factors of $(a \times k)$.

☐(b) All factors of $(b \times k)$.

☐(c) All common factors of $(a \times k)$ and $(b \times k)$.

49(a) This response is incorrect. See 49(c).

49(b) This response is incorrect. See 49(c).

49(c) This response is correct. If $k \neq 1$, then $a < (a \times k)$ and $b < (b \times k)$. Thus $\frac{a}{b}$ is a lower form of $\frac{a \times k}{b \times k}$. For example: $\frac{4}{6}$ is a lower form of $\frac{8}{12}$ since $\frac{4}{6} = \frac{4 \times 2}{6 \times 2}$ and $k = 2$.

50	$\frac{a}{b}$ is the lowest form of the fraction $\frac{a \times k}{b \times k}$ if k is a member of the set of common factors of $(a \times k)$ and $(b \times k)$ and k is the ______ (least, greatest) member in the set.	greatest
51	The set of common factors of 9 and 20 is ______.	{1}
52	The lowest form of $\frac{9}{20}$ is ______.	$\frac{9}{20}$
53	$\frac{a}{b}$ is in lowest form provided the counting number ______ is the only common factor of a and b.	1

In this sub-program, we have discovered that some fractions can be changed to equivalent fractions "in lower terms." If a fraction has no common factors in its numerator and denominator other than 1, it is said to be "in lowest terms" or "in lowest form." Any given fraction can be expressed in lowest form.

19-3. Order and Equivalence for Rational Numbers

Recalling our work with whole numbers, we see that there are essentially three relations between any two numerals m and n; either they are equivalent, that is, they name the same number; or the number n "is greater than" the number m; or the number n "is smaller than" the number m. Thus, if n and m denote members of the set of whole numbers, then one and only one of the following statements is true:

$$n = m$$

$$n > m$$

$$n < m$$

A similar statement can be made about two fractions. Given fractions $\frac{a}{b}$ and $\frac{c}{d}$, one and only one of the following statements must be true:

(1) $\frac{a}{b} = \frac{c}{d}$, that is, they are equivalent fractions;

(2) $\frac{a}{b} > \frac{c}{d}$, that is, the rational number named by the fraction $\frac{a}{b}$ is greater than the rational number named by the fraction $\frac{c}{d}$;

(3) $\frac{a}{b} < \frac{c}{d}$, that is, the rational number named by the fraction $\frac{a}{b}$ is smaller than the rational number named by the fraction $\frac{c}{d}$.

54	$\frac{a}{b} = \frac{a \times d}{b \times n}$ if $n =$ _____.	d
55	$\frac{c}{d} = \frac{c \times b}{d \times n}$ if $n =$ _____.	b
56	If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a \times d}{b \times d} = \frac{c \times n}{d \times b}$ provided $n =$ _____.	b
57	If $\frac{a \times d}{b \times d} = \frac{c \times b}{d \times b}$, then $(a \times d) = (\_\_\_ \times \_\_\_)$.	$(c \times b)$ or $(b \times c)$
58	Thus, $\frac{a}{b} = \frac{c}{d}$ if and only if $(a \times d) = (\_\_\_ \times \_\_\_)$.	$(c \times b)$ or $(b \times c)$
59	$\frac{6}{8} = \frac{3}{4}$ since $(6 \times 4) = (\_\_\_ \times \_\_\_)$.	(8×3) or (3×8)
60	$\frac{9}{12} \neq \frac{2}{3}$ since $(9 \times 3) \neq (\_\_\_ \times \_\_\_)$.	(12×2) or (2×12)
61	$\frac{2}{3} = \frac{8}{12}$ but $\frac{9}{12} \neq \frac{2}{3}$ since $(9 \times 3) \neq$ ______.	(2×12)
62	$\frac{9}{12} > \frac{8}{12}$ since $9 >$ _____.	8
63	$\frac{a}{b} = \frac{a \times d}{b \times d}$ and $\frac{c}{d} = \frac{c \times b}{d \times b}$. If $\frac{a}{b} > \frac{c}{d}$, then $\frac{a \times d}{b \times d} > \frac{c \times b}{d \times b}$ and $(a \times d) > (\_\_\_ \times \_\_\_)$.	$(c \times b)$ or $(b \times c)$

64	Thus $\frac{a}{b} > \frac{c}{d}$ if and only if $(a \times d) > (b \times \_\_\_\_)$.	c
65	$\frac{3}{4} > \frac{2}{3}$ since $(3 \times 3) > (\_\_\_ \times \_\_\_)$.	(4×2) or (2×4)
66	$\frac{7}{12} \neq \frac{2}{3}$ since $(7 \times 3) \neq (\_\_\_ \times \_\_\_)$.	(12×2) or (2×12)
67	$\frac{2}{3} = \frac{8}{12}$ but $\frac{7}{12} \neq \frac{8}{12}$ since $(7 \times 12) \neq (8 \times \_\_\_\_)$.	12
68	$\frac{7}{12} < \frac{8}{12}$ since $7 < \_\_\_\_$.	8
69	$\frac{a}{b} = \frac{a \times d}{b \times d}$ and $\frac{c}{d} = \frac{c \times b}{d \times b}$. If $\frac{a}{b} < \frac{c}{d}$, then $\frac{a \times d}{b \times d} < \frac{c \times b}{d \times b}$ and $(a \times d) < (\_\_\_ \times \_\_\_)$.	$(c \times b)$ or $(b \times c)$
70	Thus $\frac{a}{b} < \frac{c}{d}$ if and only if $(a \times d) < (b \times \_\_\_\_)$.	c
71	$\frac{4}{7} < \frac{3}{5}$ since $(4 \times 5) < (\_\_\_ \times \_\_\_)$.	(7×3) or (3×7)

72 Given fractions $\frac{a}{b}$ and $\frac{c}{d}$. Then:

☐(a) $\frac{a}{b} = \frac{c}{d}$ if and only if $a \times d = b \times c$.

☐(b) $\frac{a}{b} > \frac{c}{d}$ if and only if $a \times d > b \times c$.

☐(c) $\frac{a}{b} < \frac{c}{d}$ if and only if $a \times d < b \times c$.

All responses are correct but only one is true for a particular pair of fractions. Thus,

$\frac{2}{3} = \frac{4}{6}$ since $2 \times 6 = 3 \times 4$;

$\frac{3}{4} > \frac{2}{3}$ since $3 \times 3 > 4 \times 2$;

$\frac{3}{7} < \frac{1}{2}$ since $3 \times 2 < 7 \times 1$.

19-4. A New Property of Numbers

In the set of whole numbers, (n + 1) is called the successor of n and (n - 1) is called the predecessor of n.

73	For example, in the set of whole numbers, the successor of 4 is _____.	5
74	The successor of 19 is _____.	20
75	The successor of 215 is _____.	216
76	If n represents any whole number, then the successor of n may be represented by _______.	n + 1
77	The predecessor of 8 is _____.	7
78	The predecessor of 25 is _____.	24
79	If n represents any whole number, then the predecessor of n may be represented by _______.	n - 1
80	There are _________ (how many) whole numbers between 5 and 8?	two
81	Compute: (8 - 5) - 1 . _______	2
82	There are _________ (how many) whole numbers between 12 and 7?	four
83	Compute: (12 - 7) - 1 . _______	4
84	How many whole numbers are there between 17 and 18? _______	zero
85	Compute (18 - 17) - 1 . _______	0

86	Given whole numbers a and b such that $a < b$. If we compute $(b - a) - 1$, we have answered the question, "How many whole numbers are there between _____ and _____?"	a; b
87	There are ______________ (a finite number of, infinitely many) whole numbers between two given whole numbers a and b.	a finite number of
88	The number of whole numbers between 6 and 7 is _____.	0
89	6 is the ______ of 7 in the set of whole numbers.	predecessor
90	7 is the ______ of 6 in the set of whole numbers.	successor

91 If a and b are whole numbers such that $a > b$ and $(a - b) - 1 = 0$, then:

☐ (a) a is the successor of b.

☐ (b) b is the predecessor of a.

☐ (c) b is the successor of a.

91(a) Correct. a is greater than b and there is no whole number between a and b.

91(b) Correct. b is smaller than a and there is no whole number between a and b.

91(c) Incorrect. The successor is always greater, and $b \not> a$.

Since 7 is the successor of 6 there is no whole number between 6 and 7, and we say that 7 is "just after" 6. Since 6 is the predecessor of 7 there is no whole number between 7 and 6, and we say that 6 is "just before" 7. We say also that 6 and 7 are "next to" each other.

In general, if whole number a is the successor of whole number b, then there are no whole numbers between a and b. We say that a is "just after" b, that b is "just before" a, and that a and b are "next to" each other.

In the following frames, we consider a question concerning rational numbers: Given a rational number $\frac{a}{b}$, does it have a successor in the set of rational numbers, and does it have a predecessor in the set of rational numbers?

In the following frames, consider the two fractions $\frac{1}{2}$ and $\frac{2}{3}$.

92	The rational number represented by $\frac{1}{2}$ has many names such as $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}$ and $\frac{n}{12}$ where $n =$ _____.	6
93	The rational number represented by $\frac{2}{3}$ has many names such as $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}$ and $\frac{n}{12}$ where $n =$ _____.	8
94	Using the name $\frac{6}{12}$ for the fraction $\frac{1}{2}$ and the name $\frac{8}{12}$ for the fraction $\frac{2}{3}$, we know that $\frac{6}{12}$ is smaller than $\frac{8}{12}$ since ____ < ____.	$6 < 8$
95	Now $\frac{6}{12} < \frac{7}{12}$ since ____ < ____.	$6 < 7$
96	Hence $\frac{1}{2} < \frac{7}{12}$ since $\frac{1}{2} = \frac{n}{12}$ where $n =$ _____.	6
97	Also $\frac{7}{12} < \frac{8}{12}$ since ____ < ____.	$7 < 8$
98	Hence $\frac{7}{12} < \frac{2}{3}$ since $\frac{2}{3} = \frac{n}{12}$ where $n =$ _____.	8
99	Since $\frac{1}{2} < \frac{7}{12} < \frac{2}{3}$, we say that the fraction _______ is between $\frac{1}{2}$ and $\frac{2}{3}$.	$\frac{7}{12}$

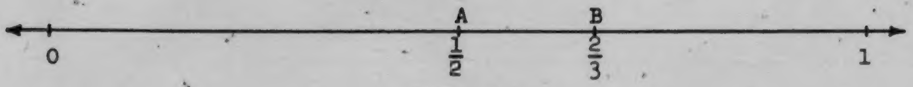
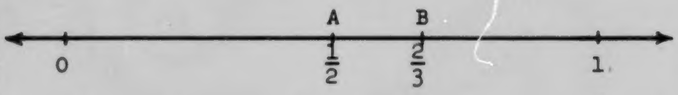
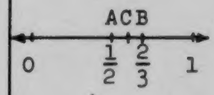


Figure 19.1

In Frames 100 - 105, refer to Figure 19.1 above.

100	Point A is labeled with the number _____.	$\frac{1}{2}$
101	A may be labeled also with the number $\frac{n}{12}$ where $n =$ _____.	6
102	Point B is labeled with the number _____.	$\frac{2}{3}$
103	B may be labeled also with the number $\frac{n}{12}$ where $n =$ _____.	8
104	Since $\frac{7}{12}$ is greater than $\frac{6}{12}$, the point C associated with the number $\frac{7}{12}$ will be to the _____ (right, left) of the point A in Figure 19.1.	right
105	Since $\frac{7}{12}$ is smaller than $\frac{8}{12}$, the point C associated with the number $\frac{7}{12}$ will be to the _____ (right, left) of the point B in Figure 19.1.	left
106	In Figure 19.2 below if point C represents the number $\frac{7}{12}$, draw C in its proper place:  Figure 19.2	

107	Since point C is between points A and B in Figure 19.2, we say the number $\frac{7}{12}$ is ______ the numbers $\frac{1}{2}$ and $\frac{2}{3}$.	between
108	$\frac{7}{12}$ is between $\frac{1}{2}$ and $\frac{2}{3}$. There ________ (is, is not) a rational number between $\frac{1}{2}$ and $\frac{7}{12}$.	is If you responded "is" to this frame, go immediately to Frame 111. If you responded "is not" continue to the next frame.
109	Another name for $\frac{1}{2}$ is $\frac{n}{24}$, where $n =$ _____.	12
110	Another name for $\frac{7}{12}$ is $\frac{n}{24}$, where $n =$ _____.	14
111	$\frac{1}{2} < \frac{n}{24} < \frac{7}{12}$ if $n =$ _____.	13
112	Hence, $\frac{n}{24}$ is between $\frac{1}{2}$ and $\frac{7}{12}$ if $n =$ _____.	13
113	It ________ (is, is not) possible to continue this process to find a number between $\frac{1}{2}$ and $\frac{13}{24}$.	is If you responded "is" to this frame, go immediately to Frame 116. If you responded "is not" continue to the next frame.
114	Another name for $\frac{1}{2}$ is $\frac{n}{48}$ where $n =$ _____.	24
115	Another name for $\frac{13}{24}$ is $\frac{n}{48}$ where $n =$ _____.	26
116	$\frac{1}{2} < \frac{n}{48} < \frac{13}{24}$ if $n =$ _____.	25

117 Hence, $\frac{n}{48}$ is between $\frac{1}{2}$ and $\frac{13}{24}$ if n = ____. — 25

118 It is possible to continue this process to find a ______ number between $\frac{1}{2}$ and $\frac{25}{48}$. — rational

119 A name for the rational number between $\frac{1}{2}$ and $\frac{25}{48}$ is $\frac{n}{96}$, where n = ____. — 49

120 Thus, there are __________________ (a finite number of, infinitely many) rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$. — infinitely many

121 In the set of rational numbers the successor of $\frac{1}{2}$ is:

☐(a) $\frac{2}{3}$

☐(b) $\frac{49}{96}$

☐(c) $\frac{2}{4}$

☐(d) $\frac{1}{2}$ does not have a successor in the set of rational numbers.

121(a) Incorrect. $\frac{13}{24}$ is between $\frac{1}{2}$ and $\frac{2}{3}$. See 121(d).

121(b) Incorrect. $\frac{97}{192}$ is between $\frac{1}{2}$ and $\frac{49}{96}$. See 121(d).

121(c) Incorrect. $\frac{1}{2}$ and $\frac{2}{4}$ are different names for the same rational number since $\frac{1}{2} = \frac{2}{4}$. See 121(d).

121(d) Correct. $\frac{1}{2}$ does not have a successor in the set of rational numbers.

122 Since $\frac{a}{b}$ has no successor in the set of rational numbers, for any rational number $\frac{c}{d}$ different from $\frac{a}{b}$ there __________ (is, is not) a rational number between $\frac{a}{b}$ and $\frac{c}{d}$. — is

123	Since there ________ (is, is not) a rational number between any two different rational numbers, the set of rational numbers is said to be dense.	is
124	Any set of numbers which has infinitely many members between two given members is said to be ______.	dense
125	If a set of numbers has two members such that there is no member between them, the set is not ______.	dense
126	The set of whole numbers ________ (is, is not) dense.	is not

127 The following sets are dense:

☐(a) the counting numbers

☐(b) the whole numbers

☐(c) the rational numbers

127(a) Incorrect. 6 and 7 are counting numbers, but there is no counting number between them.

127(b) Incorrect. 29 is a whole number, but 29 does have a whole number successor.

127(c) Correct. Between any two different rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ there is at least one other rational number and hence there are infinitely many.

In this sub-program, we have developed a new property of numbers. We have shown that between any two different rational numbers, no matter how close, there are many other rational numbers. Among other things this means that, unlike the whole numbers, one cannot identify the number that comes "just before" or "just after" a given rational number.

Chapter 20

ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

20-1. Addition of Rational Numbers

Addition of whole numbers is defined in terms of union of disjoint sets. $N(A) + N(B) = N(A \cup B)$ if and only if A and B are disjoint sets.

We use these same ideas to motivate the definition of addition of rational numbers.

The Basic Unit

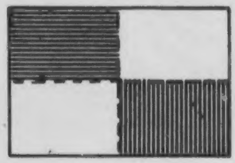
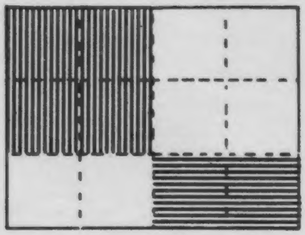


Figure 20.1

1	In Figure 20.1 above the region shaded horizontally is 1 of ______ congruent parts of the basic unit.	4
2	The fraction ______ may be used to represent the region shaded horizontally.	$\frac{1}{4}$
3	In Figure 20.1 the region shaded vertically is 1 of ______ congruent parts of the basic unit.	4
4	The fraction ______ may be used to represent the region shaded vertically.	$\frac{1}{4}$
5	In Figure 20.1 the union of the shaded regions is ______ of the 4 congruent regions of the basic unit.	2

6	The fraction ______ may be used to represent the union of the shaded regions.	$\frac{2}{4}$
7	Since the shaded regions represent disjoint sets, their union is represented by $\frac{2}{4}$ and also can be represented by $\underline{\frac{1}{4} + \text{---}}$.	$\frac{1}{4} + \frac{1}{4}$
8	Figure 20.2 is a model for $\frac{4}{12} + \frac{2}{12} = \frac{\quad}{12}$.  Figure 20.2	6
9	Since $6 = 4 + 2$, $\frac{4}{12} + \frac{2}{12} = \frac{(\quad + \quad)}{12} = \frac{6}{12}$.	$(4 + 2)$

From these examples we can motivate the definition of addition of rational numbers as follows: The union of a of the b congruent parts of a unit region and c of the b congruent parts of the unit region is the same as (a + c) of the b congruent parts of the unit region, when the two subregions are disjoint.

Thus, let us make the following definition of addition of a pair of rational numbers having the same denominators:

Definition: Given two fractions $\frac{a}{b}$ and $\frac{c}{b}$. Then $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$.

10 $\frac{3}{4} + \frac{5}{4}$ is the same as

☐ (a) $\frac{8}{8}$ ☐ (b) $\frac{8}{4}$ ☐ (c) $\frac{4}{2}$ ☐ (d) $\frac{3+5}{4}$

10(a) Incorrect since $\frac{3}{4} + \frac{5}{4} = \frac{3+5}{4}$, not $\frac{3+5}{8}$.

10(b) Correct since $3 + 5 = 8$. 10(c) and 10(d) also are correct.

10(c) Correct since $3 + 5 = 8$ and $\frac{8}{4} = \frac{4}{2}$. 10(b) and 10(d) also are correct.

10(d) Correct by the definition for addition of rational numbers. 10(b) and 10(c) also are correct.

20-2. Properties of Addition of Rational Numbers

We should now check to see whether or not addition of rational numbers as we have defined it has the properties characteristic of addition of whole numbers. These properties are : (1) closure; (2) commutativity; (3) associativity; (4) additive identity.

11	If a and c are whole numbers, then $(a + c)$ is a ______ number.	whole
12	Hence, the set of whole numbers is ______ under the operation of addition.	closed
13	If a, b and c are whole numbers and $b \neq 0$, then $\frac{a+c}{b}$ ______ (is, is not) a rational number.	is
14	Since $\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b}$, then $\frac{a}{b} + \frac{c}{b}$ ______ (is, is not) a rational number.	is
15	Thus, $\frac{a}{b} + \frac{c}{b}$ is always a rational number and the set of rational numbers is ______ under the operation of addition.	closed

We have seen that the set of rational numbers with the same denominators is closed under the operation of addition.

We now turn to another property of the whole numbers to see if it applies also to the rational numbers.

16	$\frac{2}{5} + \frac{1}{5}$ ________ (is, is not) equal to $\frac{1}{5} + \frac{2}{5}$.	is
17	$\frac{4}{7} + \frac{3}{7}$ is ________ (equal, not equal) to $\frac{3}{7} + \frac{4}{7}$.	equal
18	The order of the addends in the sum of two rational numbers ________ (does, does not) give different results.	does not
19	The addition of two rational numbers is independent of the ______ in which they are added.	order
20	The two examples given in Frames 16 and 17 suggest the conclusion that addition of rational numbers has the ______ property.	commutative
21	A finite number of examples ________ (is, is not) sufficient to draw a general conclusion.	is not

A finite number of examples can give an intuitive justification for a generalization, but the following theorem and proof furnish conclusive evidence that the rational numbers are commutative under addition.

Theorem:

$\frac{r}{s} + \frac{t}{s} = \frac{t}{s} + \frac{r}{s}$, if $\frac{r}{s}$ and $\frac{t}{s}$ are rational numbers.

Proof:

22	$\frac{r}{s} + \frac{t}{s} = \frac{r+t}{s}$, by the definition of ______ of rational numbers.	addition

23	$\frac{r + t}{s} = \frac{t + r}{s}$ since addition of whole numbers is _______.	commutative
24	$\frac{t + r}{s} = \frac{t}{s} + \frac{r}{s}$ by the definition of addition of _______ numbers.	rational
25	Therefore, $\frac{r}{s} + \frac{t}{s} = \frac{t}{s} + \frac{r}{s}$ is $\underset{\text{(true, not true)}}{\underline{\hspace{3cm}}}$ for any pair of rational numbers.	true

We have seen that the sum of two numbers is independent of the order of the addends, for both rational numbers and whole numbers. That is, addition in the set of rational numbers has the commutative property.

Let us see if the result of performing two or more successive additions is independent of the order in which the additions are performed.

26	$(\frac{2}{7} + \frac{3}{7}) + \frac{8}{7} = \frac{5}{7} + \frac{8}{7} = \frac{\underline{\hspace{1cm}}}{7}$.	$\frac{13}{7}$
27	$\frac{2}{7} + (\frac{3}{7} + \frac{8}{7}) = \frac{2}{7} + \frac{11}{7} = \frac{\underline{\hspace{1cm}}}{7}$.	$\frac{13}{7}$
28	$(\frac{2}{7} + \frac{3}{7}) + \frac{8}{7} \underset{(=,\ \neq)}{\underline{\hspace{1.5cm}}} \frac{2}{7} + (\frac{3}{7} + \frac{8}{7})$.	=
29	The three preceding frames seem to indicate that the sum of three rational numbers $\underset{\text{(is, is not)}}{\underline{\hspace{3cm}}}$ independent of the order of performing the additions.	is
30	$(\frac{a}{d} + \frac{b}{d}) + \frac{c}{d} = \frac{(a + b)}{d} + \frac{c}{d}$ by the definition of addition of _______ numbers.	rational
31	$\frac{(a + b)}{d} + \frac{c}{d} = \frac{(a + b) + c}{d}$ by the definition of _______ of rational numbers.	addition
32	$\frac{(a + b) + c}{d} = \frac{a + (b + c)}{d}$ by the _______ property of whole numbers.	associative

33	$\frac{a + (b + c)}{d} = \frac{a}{d} + \frac{(b + c)}{d}$ by the definition of addition of _______ numbers.	rational
34	$\frac{a}{d} + \frac{(b + c)}{d} = \frac{a}{d} + (\frac{b}{d} + \frac{c}{d})$ by the definition of _______ of rational numbers.	addition
35	Therefore, $(\frac{a}{d} + \frac{b}{d}) + \frac{c}{d} \underset{(=,\ \neq)}{\underline{\qquad}} \frac{a}{d} + (\frac{b}{d} + \frac{c}{d})$.	=

36 The statement $(\frac{a}{d} + \frac{b}{d}) + \frac{c}{d} = \frac{a}{d} + (\frac{b}{d} + \frac{c}{d})$ shows symbolically that addition of rational numbers has

☐ (a) the closure property

☐ (b) the commutative property

☐ (c) the associative property

36(a) Incorrect. While addition in the set of rational numbers has the closure property, the statement is a symbolic representation of the associative property.

36(b) Incorrect. While addition in the set of rational numbers has the commutative property, the statement is a symbolic representation of the associative property.

36(c) Correct. The statement

$$(\frac{a}{d} + \frac{b}{d}) + \frac{c}{d} = \frac{a}{d} + (\frac{b}{d} + \frac{c}{d})$$

indicates that the result of performing two successive additions is independent of their order.

We have seen that the result of performing two successive additions is independent of the order in which the additions are performed. That is, addition in the set of rational numbers has the associative property. It is possible to verify that the result of performing any finite number of successive additions is independent of the order in which the additions are performed.

Zero is the identity element for addition in the set of whole numbers. That is, $0 + n = n + 0 = n$ for any whole number n.

37 The set of rational numbers ______ (does, does not) have an identity element for addition.

does

38 The identity for addition in the set of rational numbers is:

☐ (a) $\frac{1}{0}$ ☐ (b) $\frac{0}{n}$ ☐ (c) $\frac{0}{0}$

38(a) Incorrect. $\frac{1}{0}$ does not represent a rational number.

38(b) Correct, provided n is a counting number.
Proceed to Frame 42.

38(c) Incorrect. $\frac{0}{0}$ does not represent a rational number.

39 $\frac{3}{7} + \frac{0}{7} = \frac{\quad}{7}$.

3 + 0 or 3

40 $\frac{5}{9} + \frac{0}{9} = \frac{\quad}{9}$.

5 + 0 or 5

41 $\frac{7}{4} + \frac{0}{n} = \frac{7}{4}$ if $n =$ ______.

4

42 If $b \neq 0$, $\frac{a}{b} + \frac{0}{b} = \frac{a + 0}{b}$ by the definition of ______ of rational numbers.

addition

43 If $b \neq 0$, $\frac{a + 0}{b} = \frac{a}{b}$ since $a + 0 =$ ______. a

44 Hence, $\frac{a}{b} + \frac{0}{b} = \frac{a}{b}$, and $\frac{0}{b}$, where $b \neq 0$, is the identity for ______ of rational numbers. addition

We have defined addition of fractions for the case where the denominators are the same. That is, $\frac{a}{d} + \frac{c}{d} = \frac{a + c}{d}$, provided $d \neq 0$. If the denominators are not the same, we use the idea of equivalent fractions (as discussed in Chapter 19) to arrive at a suitable definition for addition of rational numbers.

45 The sum of $\frac{1}{2}$ and $\frac{1}{3}$ is

☐ (a) $\frac{3 + 2}{6}$

☐ (b) $\frac{2}{5}$

☐ (c) $\frac{1}{6}$

45(a) Correct. $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{3 + 2}{6}$.

45(b) Incorrect. $\frac{1}{2} + \frac{1}{3} \neq \frac{1 + 1}{2 + 3}$.

45(c) Incorrect. $\frac{1}{6}$ is the product (or the difference) of $\frac{1}{2}$ and $\frac{1}{3}$, not the sum.
See 45(a).

46 Since $\frac{a}{b} = \frac{a \times d}{b \times d}$ and $\frac{c}{d} = \frac{c \times b}{d \times b}$,

$\frac{a}{b} + \frac{c}{d} = \frac{a \times d}{b \times d} + \frac{(\quad \times \quad)}{d \times b}$. (c × b)

47	Since $(b \times d) = (d \times b)$, the fractions $\frac{a \times d}{b \times d}$ and $\frac{c \times b}{d \times b}$ have $\underset{\text{(the same, different)}}{\underline{\qquad\qquad\qquad}}$ denominators.	the same
48	And $\frac{a \times d}{b \times d} + \frac{c \times b}{d \times b} = \frac{(a \times d) + (c \times b)}{b \times d}$ by the definition of ______ rational numbers.	addition
49	Hence, $\frac{a}{b} + \frac{c}{d} = \frac{(\quad \times \quad) + (\quad \times \quad)}{b \times d}$.	$(a \times d) + (b \times c)$
50	Thus, $\frac{2}{3} + \frac{5}{7} = \frac{(2 \times \quad)}{3 \times 7} + \frac{(5 \times \quad)}{7 \times 3}$.	7; 3
51	$\frac{2}{3} + \frac{5}{7} = \frac{(\quad \times \quad) + (\quad \times \quad)}{3 \times 7}$.	$(2 \times 7) + (5 \times 3)$
52	$\frac{2}{3} + \frac{5}{7} = \frac{(\quad + \quad)}{21}$.	$14 + 15$
53	$\frac{2}{3} + \frac{5}{7} = \frac{\quad}{\quad}$.	$\frac{29}{21}$

54 The sum of $\frac{17}{12}$ and $\frac{5}{26}$ is

☐ (a) $\frac{22}{38}$ ☐ (b) $\frac{502}{312}$ ☐ (c) $\frac{251}{156}$

54(a) Incorrect since $\frac{17}{12} + \frac{5}{26} \neq \frac{17 + 5}{12 + 26}$.

54(b) Correct since $\frac{17}{12} + \frac{5}{26} = \frac{(17 \times 26) + (12 \times 5)}{12 \times 26} = \frac{442 + 60}{312} = \frac{502}{312}$.

54(c) also is correct.

54(c) Correct since $\frac{17}{12} + \frac{5}{26} =$

$$\frac{(17 \times 26) + (12 \times 5)}{12 \times 26} = \frac{442 + 60}{312} = \frac{502}{312} = \frac{251 \times 2}{156 \times 2} = \frac{251}{156}.$$

54(b) also is correct.

20-3. Subtraction of Rational Numbers

In the set of whole numbers, subtraction may be defined as follows:

Definition: a - c = n if and only if n + c = a.

We wish to define subtraction in the set of rational numbers in an analogous manner, as follows:

Definition: $\frac{a}{d} - \frac{c}{d} = \frac{n}{d}$ if and only if $\frac{n}{d} + \frac{c}{d} = \frac{a}{d}$.

55	$\frac{3}{7} - \frac{2}{7} = \frac{\_\_\_}{7}$ since $\frac{1}{7} + \frac{2}{7} = \frac{3}{7}$.	1
56	$\frac{7}{9} - \frac{3}{9} = \frac{\_\_\_}{9}$ since $\frac{4}{9} + \frac{3}{9} = \frac{7}{9}$.	4
57	$\frac{n}{d} + \frac{c}{d} = \frac{n + c}{d}$ by the definition of ______ of rational numbers.	addition

58	If $\frac{n}{d} + \frac{c}{d} = \frac{a}{d}$, then $\frac{n + c}{d} = \frac{\underline{\qquad}}{d}$.	a
59	If $\frac{n + c}{d} = \frac{a}{d}$, then $n + c = \underline{\qquad}$.	a
60	If $n + c = a$, then $n = a - \underline{\qquad}$.	a - c
61	Thus, if $\frac{n}{d} + \frac{c}{d} = \frac{a}{d}$, then $n = a - \underline{\qquad}$.	a - c
62	If $\frac{n}{d} + \frac{c}{d} = \frac{a}{d}$, then $\frac{a}{d} - \frac{c}{d} = \frac{n}{d}$ and $n = a - c$. Therefore, $\frac{a}{d} - \frac{c}{d} = \frac{(a - \qquad)}{d}$.	c
63	$\frac{7}{5} - \frac{4}{5} = \frac{(\qquad - \qquad)}{5} = \frac{\underline{\qquad}}{5}$.	(7 - 4); 3
64	$\frac{a}{b} - \frac{c}{d} = \frac{a \times d}{b \times d} - \frac{c \times \qquad}{d \times b}$.	b
65	$\frac{a \times d}{b \times d} - \frac{c \times b}{d \times b} = \frac{(a \times d) - (\qquad \times \qquad)}{b \times d}$.	(b × c) or (c × b)
66	Hence, $\frac{a}{b} - \frac{c}{d} = \frac{(\qquad) - (\qquad)}{b \times d}$.	(a × d) - (b × c)
67	$\frac{4}{5} - \frac{3}{7} = \frac{(4 \times 7)}{5 \times 7} - \frac{(\qquad \times \qquad)}{7 \times 5}$.	(3 × 5) or (5 × 3)
68	$\frac{4 \times 7}{5 \times 7} - \frac{3 \times 5}{7 \times 5} = \frac{28 - \qquad}{35}$.	15
69	Hence, $\frac{4}{5} - \frac{3}{7} = \frac{28 - 15}{35} = \frac{\underline{\qquad}}{35}$.	13
70	$\frac{1}{2} - \frac{1}{3} = \frac{3 - 2}{6} = \frac{\underline{\qquad}}{\underline{\qquad}}$.	$\frac{1}{6}$
71	$\frac{4}{5} - \frac{1}{2} = \frac{8 - 5}{10} = \frac{\underline{\qquad}}{\underline{\qquad}}$.	$\frac{3}{10}$

20-4. Properties of Subtraction

72	$\frac{3}{7} - \frac{5}{7} = \frac{(\quad - \quad)}{7}$.	(3 - 5)
73	But (3 - 5) $\underset{\text{(does, does not)}}{\underline{\qquad\qquad}}$ represent a whole number.	does not
74	Hence, subtraction of rational numbers $\underset{\text{(does, does not)}}{\underline{\qquad\qquad}}$ have the closure property.	does not
75	$\frac{3}{7} - \frac{5}{7} \underset{(=,\ \neq)}{\underline{\qquad}} \frac{5}{7} - \frac{3}{7}$.	$\neq$
76	Hence, the set of rational numbers under the operation of subtraction $\underset{\text{(is, is not)}}{\underline{\qquad\qquad}}$ commutative.	is not
77	$(\frac{9}{7} - \frac{5}{7}) - \frac{3}{7} = \frac{4}{7} - \frac{3}{7} = \underline{\qquad}$.	$\frac{1}{7}$
78	$\frac{9}{7} - (\frac{5}{7} - \frac{3}{7}) = \frac{9}{7} - \frac{2}{7} = \underline{\qquad}$.	$\frac{7}{7}$
79	Hence, $(\frac{9}{7} - \frac{5}{7}) - \frac{3}{7} \neq \frac{9}{7} - (\frac{5}{7} - \frac{3}{7})$ and the rational numbers $\underset{\text{(do, do not)}}{\underline{\qquad\qquad}}$ have the associative property under the operation of subtraction.	do not
80	$\frac{5}{3} - \frac{5}{3} = \frac{5 - 5}{3} = \frac{\quad}{3}$.	0
81	$\frac{a}{b} - \frac{a}{b} = \frac{a - a}{b} = \frac{\quad}{b}$.	0
82	$\frac{7}{4} - \frac{0}{4} = \frac{7 - 0}{4} = \frac{\quad}{4}$.	7
83	$\frac{a}{b} - \frac{0}{b} = \frac{a - 0}{b} = \frac{\quad}{b}$.	a

84	$(\frac{10}{13} + \frac{6}{13}) - \frac{6}{13} = \frac{16}{13} - \frac{6}{13} = \underline{\qquad}$.	$\frac{10}{13}$
85	$(\frac{a}{b} + \frac{c}{b}) - \frac{c}{b} = \frac{a + c}{b} - \frac{c}{b}$ by the definition of ______ of rational numbers.	addition
86	$\frac{a + c}{b} - \frac{c}{b} = \frac{(a + c) - c}{b}$ by subtraction of ______ numbers.	rational
87	$\frac{(a + c) - c}{b} = \frac{\quad}{b}$.	a
88	Hence, $(\frac{a}{b} + \frac{c}{b}) - \frac{c}{b} = \frac{\quad}{b}$.	a
89	$(\frac{10}{13} - \frac{5}{13}) + \frac{5}{13} + \frac{5}{13} = \frac{\quad}{13}$.	10
90	$(\frac{a}{b} - \frac{c}{b}) + \frac{c}{b} = \frac{a - c}{b} + \frac{c}{b}$ by subtraction of ______ numbers.	rational
91	$\frac{a - c}{b} + \frac{c}{b} = \frac{(a - c) + c}{b}$ by the definition for ______ of rational numbers.	addition
92	$\frac{(a - c) + c}{b} = \frac{\quad}{b}$.	a
93	Hence, $(\frac{a}{b} - \frac{c}{b}) + \frac{c}{b} = \frac{\quad}{b}$.	a
94	$(\frac{9}{7} - \frac{4}{7}) + \frac{4}{7} = \frac{5}{7} + \frac{4}{7} = \frac{\quad}{7}$.	9
95	$(\frac{a}{d} - \frac{c}{d}) + \frac{c}{d} \underset{(=,\ \neq)}{\underline{\qquad}} (\frac{a}{d} + \frac{c}{d}) - \frac{c}{d}$.	=
96	$(\frac{7}{4} - \frac{3}{4}) + \frac{3}{4} = (\frac{7}{4} + \frac{3}{4}) - \underline{\qquad}$.	$\frac{3}{4}$

Chapter 21

MULTIPLICATION OF RATIONAL NUMBERS

21-1. Multiplication of Rational Numbers

In the last chapter we defined addition for rational numbers in a way that used only properties of whole numbers. We showed that this addition has such properties as closure, commutativity and associativity that are characteristic of addition of whole numbers. We found that the binary operation addition in the set of rational numbers involves taking two numbers and associating with them a third number called their "sum."

Our tasks for multiplication of rational numbers are clearly of the same sort. With each pair of rational numbers we want to associate a third number called their "product." We want ways of doing this that involve only previously learned operations and concepts. We would like the properties of such multiplication to be pretty much the same as those of the now familiar multiplication of whole numbers. Furthermore, in order to be consistent with our efforts so far, we want to find physical models which justify and give content to the procedures we develop.

We define multiplication of rational numbers as follows:

Definition: Given two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$. Then $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

This definition gives a computational procedure that depends only on multiplication of whole numbers. As with whole numbers, we call $\frac{a \times c}{b \times d}$ the product of the two factors $\frac{a}{b}$ and $\frac{c}{d}$.

1	$\frac{2}{3} \times \frac{4}{7} = \frac{2 \times 4}{3 \times 7} = \underline{\qquad}$.	$\frac{8}{21}$
2	$\frac{4}{7} \times \frac{2}{3} = \frac{4 \times 2}{7 \times 3} = \underline{\qquad}$.	$\frac{8}{21}$
3	$\frac{3}{5} \times \frac{4}{1} = \frac{(\quad \times \quad)}{5 \times 1} = \frac{12}{5}$.	(3 × 4)
4	$\frac{7}{6} \times \frac{5}{5} = \frac{(\quad \times \quad)}{6 \times 5} = \frac{35}{30}$.	(7 × 5)

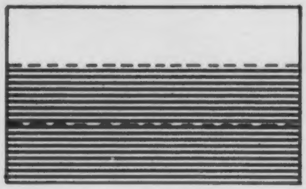
5 $\frac{4}{4} \times \frac{7}{6} = \frac{4 \times 7}{4 \times 6} =$ ______. $\frac{28}{24}$

6 $\frac{4}{5} \times \frac{0}{3} = \frac{4 \times 0}{5 \times 3} =$ ______. $\frac{0}{15}$

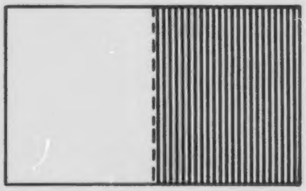
7 $\frac{0}{3} \times \frac{4}{5} = \frac{0 \times 4}{3 \times 5} =$ ______. $\frac{0}{15}$

The following frames attempt to exhibit some models illustrating the product of two rational numbers.

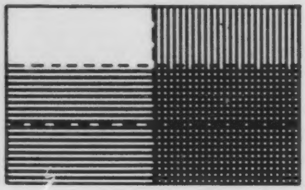
8 Consider the figure below. The shaded region is a model for the rational number ______. $\frac{2}{3}$



9 Consider the figure below. The shaded region is a model for the rational number ______. $\frac{1}{2}$

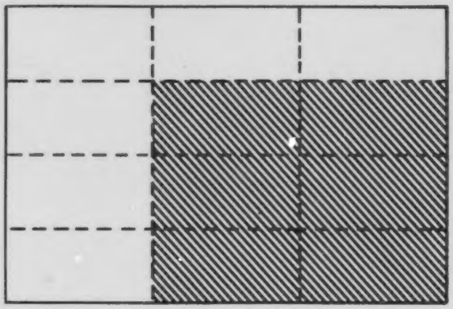


10 Consider the figure below. The cross-hatched region is a model for the rational number $\frac{2}{6}$ or $\underline{\;\text{—} \times \text{—}\;}$.



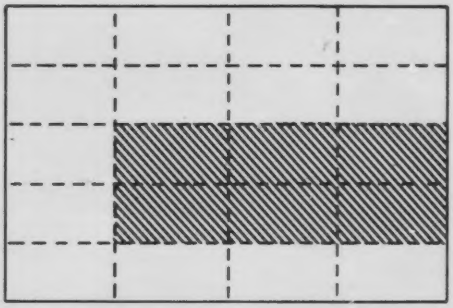
$\frac{2}{3} \times \frac{1}{2}$

11 Consider the figure below. The shaded region represents the rational number $\frac{6}{12}$ or $\underline{\frac{3}{4} \times \text{—}}$.



$\frac{3}{4} \times \frac{2}{3}$

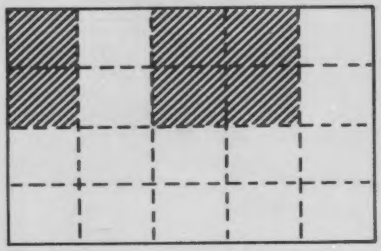
12 Consider the figure below. The shaded region represents the rational number ______ or $\frac{2}{5} \times \frac{3}{4}$.



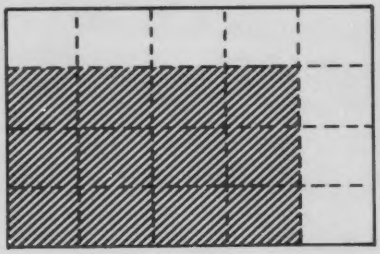
$\frac{6}{20}$

13 Which of the following is a model for $\frac{2}{4} \times \frac{3}{5}$? (Check all correct responses.)

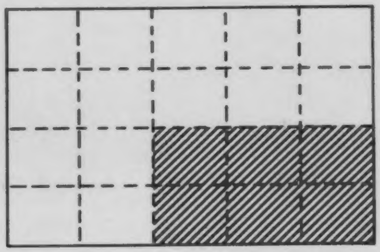
☐ (a)



☐ (b)



☐ (c)



13(a) Correct. $\frac{2}{4} \times \frac{3}{5} = \frac{2 \times 3}{4 \times 5} = \frac{6}{20}$. See 13(c).

13(b) Incorrect. $\frac{2}{4} \times \frac{3}{5} \neq \frac{12}{20}$.

13(c) Correct. $\frac{2}{4} \times \frac{3}{5} = \frac{2 \times 3}{4 \times 5} = \frac{6}{20}$. See 13(a).

21-2. Properties of Multiplication of Rational Numbers

We should now check to see whether or not multiplication of rational numbers as we have defined it has the properties characteristic of multiplication of whole numbers. These properties are: (1) closure; (2) commutativity; (3) associativity; (4) multiplicative identity.

14	If a and b are whole numbers, then $(a \times b)$ is a _______ number.	whole
15	Hence, the set of whole numbers is _______ under the operation of multiplication.	closed
16	If a, b, c and d are whole numbers, $b \neq 0$ and $d \neq 0$, then $\frac{a \times c}{b \times d}$ $\underset{\text{(is, is not)}}{\underline{\qquad\qquad}}$ a rational number.	is
17	Since $\frac{a \times c}{b \times d} = \frac{a}{b} \times \frac{c}{d}$, then $\frac{a}{b} \times \frac{c}{d}$ $\underset{\text{(is, is not)}}{\underline{\qquad\qquad}}$ a rational number.	is
18	Thus, if $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} \times \frac{c}{d}$ is always a rational number, and the set of rational numbers is _______ under the operation of multiplication.	closed

We have seen that the set of rational numbers is closed under the operation of multiplication.

We now turn to another property of the whole numbers to see if it applies also to multiplication of rational numbers.

19	$\frac{4}{5} \times \frac{2}{3}$ $\underset{\text{(is, is not)}}{\underline{\qquad\qquad}}$ equal to $\frac{2}{3} \times \frac{4}{5}$.	is
20	$\frac{2}{7} \times \frac{5}{3}$ is $\underset{\text{(equal, not equal)}}{\underline{\qquad\qquad\qquad}}$ to $\frac{5}{3} \times \frac{2}{7}$.	equal

21	The order of the factors in the product of two rational numbers ______ (does, does not) give different results.	does not
22	The multiplication of two rational numbers is independent of the ______ in which they are multiplied.	order
23	The two examples given in Frames 19 and 20 suggest the conclusion that multiplication of rational numbers has the ______ property.	commutative
24	A finite number of examples ______ (is, is not) sufficient to draw a general conclusion.	is not

A finite number of examples can give an intuitive justification for a generalization, but the following theorem and proof furnish conclusive evidence that the rational numbers are commutative under multiplication.

	Theorem: $\frac{r}{s} \times \frac{t}{v} = \frac{t}{v} \times \frac{r}{s}$, if $\frac{r}{s}$ and $\frac{t}{v}$ are rational numbers.	
	Proof:	
25	$\frac{r}{s} + \frac{t}{v} = \frac{r \times t}{s \times v}$ by the definition of ______ of rational numbers.	multiplication
26	$\frac{r \times t}{s \times v} = \frac{t \times r}{v \times s}$ since multiplication of whole numbers is ______.	commutative
27	$\frac{t \times r}{v \times s} = \frac{t}{v} \times \frac{r}{s}$ by the definition of multiplication of ______ numbers.	rational
28	Therefore, $\frac{r}{s} \times \frac{t}{v} = \frac{t}{v} \times \frac{r}{s}$ is ______ (true, not true) for any pair of rational numbers.	true

We have seen that the product of two numbers is independent of the order of the factors for both rational numbers and whole numbers. That is, multiplication in the set of rational numbers has the commutative property.

Let us see if the result of performing two or more successive multiplications is independent of the order in which the multiplications are performed.

29	$(\frac{1}{2}\times\frac{3}{5})\times\frac{7}{4}=\frac{3}{10}\times\frac{7}{4}=\frac{\underline{\qquad}}{40}$.	$\frac{21}{40}$
30	$\frac{1}{2}\times(\frac{3}{5}\times\frac{7}{4})=\frac{1}{2}\times\frac{21}{20}=\frac{\underline{\qquad}}{40}$.	$\frac{21}{40}$
31	$(\frac{1}{2}\times\frac{3}{5})\times\frac{7}{4}\ \underset{(=,\ \neq)}{\underline{\qquad}}\ \frac{1}{2}\times(\frac{3}{5}\times\frac{7}{4})$.	=
32	The three preceding frames seem to indicate that the product of three rational numbers $\underset{\text{(is, is not)}}{\underline{\qquad\qquad}}$ independent of the order of performing the multiplications.	is
33	$(\frac{a}{b}\times\frac{c}{d})\times\frac{e}{f}=\frac{(a\times c)}{(b\times d)}\times\frac{e}{f}$ by the definition of multiplication of ______ numbers.	rational
34	$\frac{(a\times c)}{(b\times d)}\times\frac{e}{f}=\frac{(a\times c)\times e}{(b\times d)\times f}$ by the definition of ______ of rational numbers.	multiplication
35	$\frac{(a\times c)\times e}{(b\times d)\times f}=\frac{a\times(c\times e)}{b\times(d\times f)}$ by the ______ property of multiplication of whole numbers.	associative
36	$\frac{a\times(c\times e)}{b\times(d\times f)}=\frac{a}{b}\times\frac{(c\times e)}{(d\times f)}$ by the definition of multiplication of ______ numbers.	rational
37	$\frac{a}{b}\times\frac{(c\times e)}{(d\times f)}=\frac{a}{b}\times(\frac{c}{d}\times\frac{e}{f})$ by the definition of ______ of rational numbers.	multiplication
38	Therefore, $(\frac{a}{b}\times\frac{c}{d})\times\frac{e}{f}\ \underset{(=,\ \neq)}{\underline{\qquad}}\ \frac{a}{b}\times(\frac{c}{d}\times\frac{e}{f})$.	=

39 The statement $(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$ shows symbolically that multiplication of rational numbers has

☐ (a) the closure property

☐ (b) the commutative property

☐ (c) the associative property

39(a) Incorrect. While multiplication in the set of rational numbers has the closure property, the statement is a symbolic representation of the associative property.

39(b) Incorrect. While multiplication in the set of rational numbers has the commutative property, the statement is a symbolic representation of the associative property.

39(c) Correct. The statement

$$(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$$

indicates that the result of performing two successive multiplications is independent of their order.

We have seen that the result of performing two successive multiplications is independent of the order in which the multiplications are performed. That is, multiplication in the set of rational numbers has the associative property. It is possible to verify that the result of performing any finite number of successive multiplications is independent of the order in which the multiplications are performed.

One is the identity element for multiplication in the set of whole numbers. That is, $1 \times n = n \times 1 = n$ for any whole number $\underline{n}$.

40 The identity for multiplication in the set of rational numbers is represented by the fraction:

☐ (a) $\frac{0}{n}$ ☐ (b) $\frac{n}{n}$ ☐ (c) $\frac{1}{1}$

40(a) Incorrect. $\frac{0}{n}$ is the identity for addition, not multiplication.

40(b) Correct, provided $n \neq 0$. Proceed to Frame 45.

40(c) Correct. $\frac{1}{1}$ is another name for $\frac{n}{n}$. See 40(b) and proceed to Frame 45.

41	$\frac{2}{5} \times \frac{1}{1} = \frac{2 \times 1}{5 \times 1} =$ ______.	$\frac{2}{5}$
42	$\frac{2}{5} \times \frac{2}{2} = \frac{4}{10} = \frac{n}{5}$ where $n =$ ______.	2
43	$\frac{2}{5} \times \frac{7}{7} = \frac{14}{35} = \frac{2}{n}$ where $n =$ ______.	5
44	$\frac{2}{5} \times \frac{n}{n} = \frac{2 \times n}{5 \times n} = \frac{\underline{\quad}}{5}$.	2
45	$\frac{a}{b} \times \frac{n}{n} = \frac{a \times n}{b \times n}$ by definition of ______ of rational numbers.	multiplication
46	But $\frac{a \times n}{b \times n}$ $\underset{\text{(is, is not)}}{\underline{\qquad\qquad}}$ equivalent to $\frac{a}{b}$.	is
47	Hence, $\frac{a}{b} \times \frac{n}{n} = \frac{\underline{\quad}}{b}$, provided $n \neq 0$.	a
48	Since $\frac{a}{b} \times \frac{n}{n} = \frac{a}{b}$, the identity for ______ of rational numbers is $\frac{n}{n} = \frac{1}{1}$.	multiplication
49	The set of rational numbers $\underset{\text{(does, does not)}}{\underline{\qquad\qquad}}$ have an element which is the identity for multiplication.	does

Recall that multiplication is distributive over addition in the set of whole numbers. That is, $a \times (b + c) = (a \times b) + (a \times c)$.

Let us see if multiplication is distributive over addition in the set of rational numbers.

Theorem:

$$\frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{d}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{d}\right),$$

if $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{d}$ are rational numbers.

Proof:

50	$\frac{a}{b} \times (\frac{c}{d} + \frac{e}{d}) = \frac{a}{b} \times \frac{(c + e)}{d}$ by the definition of _____ of rational numbers, and $d \neq 0$.	addition
51	$\frac{a}{b} \times \frac{(c + e)}{d} = \frac{a \times (c + e)}{b \times d}$ by the definition of _____ of rational numbers.	multiplication
52	$\frac{a \times (c + e)}{b \times d} = \frac{(a \times c) + (a \times e)}{b \times d}$ by the _____ property of multiplication over addition in the set of whole numbers.	distributive
53	$\frac{(a \times c) + (a \times e)}{b \times d} = \frac{(a \times c)}{b \times d} + \frac{(a \times e)}{b \times d}$ by the definition of addition of _____ numbers.	rational
54	$\frac{(a \times c)}{b \times d} + \frac{(a \times e)}{b \times d} = (\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{d})$ by the definition of multiplication of _____ numbers.	rational
55	Therefore, $\frac{a}{b} \times (\frac{c}{d} + \frac{e}{d}) = (\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{d})$ is _____ (true, not true) for any three rational numbers.	true
56	Hence, in the set of rational numbers, multiplication is _____ over addition.	distributive

57 $\frac{1}{2} \times (\frac{3}{4} + \frac{1}{4}) = \frac{1}{2} \times \frac{\underline{\quad}}{4} = \frac{\underline{\quad}}{8}$. 4; 4

58 $\frac{1}{2} \times (\frac{3}{4} + \frac{1}{4}) = (\frac{1}{2} \times \frac{3}{4}) + (\frac{1}{2} \times \frac{1}{4}) = \frac{\underline{\quad}}{8} + \frac{\underline{\quad}}{8} = \frac{4}{8}$. 3; 1

59 $\frac{2}{3} \times (\frac{1}{8} + \frac{4}{8}) = \frac{\underline{\quad}}{24} + \frac{8}{24} = \frac{\underline{\quad}}{24}$. 2; 10

60 $\frac{3}{5} \times (\frac{1}{2} + \frac{1}{6}) = \frac{3}{5} \times (\frac{\underline{\quad}}{6} + \frac{1}{6}) = \frac{3}{5} \times \frac{4}{6} = \frac{\underline{\quad}}{30}$. 3; 12

61 $\frac{3}{5} \times (\frac{1}{2} + \frac{1}{6}) = (\frac{3}{5} \times \frac{1}{2}) + (\frac{3}{5} \times \frac{1}{6})$
$= \frac{3}{10} + \frac{3}{30} = \frac{\underline{\quad}}{30} + \frac{3}{30} = \frac{\underline{\quad}}{30}$. 9; 12

A more general representation of the distributive property is the statement:

$$\frac{a}{b} \times (\frac{c}{d} + \frac{e}{f}) = (\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{f}).$$

62 $\frac{2}{3} \times (\frac{3}{2} + \frac{4}{5}) = \frac{6}{6} + \frac{\underline{\quad}}{15}$. 8

63 $\frac{3}{5} \times (\frac{1}{2} + \frac{6}{7}) = \frac{\underline{\quad}}{10} + \frac{\underline{\quad}}{35}$. 3; 18

In the set of whole numbers and rational numbers the identity element for addition, denoted by 0, has the following multiplication property $0 \times n = n \times 0 = 0$ for any whole number n.

64	$\frac{3}{7} \times \frac{0}{2} = \frac{3 \times 0}{7 \times 2} = \frac{\_\_\_\_}{14}$.	0
65	$\frac{5}{8} \times \frac{0}{3} = \frac{5 \times 0}{8 \times 3} = \frac{\_\_\_\_}{24}$.	0
66	$\frac{a}{b} \times \frac{0}{n} = \frac{a \times 0}{b \times n}$ by the definition of ______ of rational numbers.	multiplication
67	$\frac{a \times 0}{b \times n} = \frac{0}{k}$ since $a \times 0 =$ ______.	0
68	Hence, $\frac{a}{b} \times \frac{0}{n} = \frac{0}{k} = 0$ and the product of $\frac{a}{b}$ and the identity for addition in rational numbers is equal to ______.	$\frac{0}{n}$ or 0
69	$\frac{2}{3} \times \frac{3}{2} = \frac{2 \times 3}{3 \times 2} = \frac{n}{6}$ where $n =$ ______.	6
70	$\frac{1}{5} \times \frac{5}{1} = \frac{1 \times 5}{5 \times 1} = \frac{5}{n}$ where $n =$ ______.	5
71	$\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a} = \frac{n}{n} = \frac{1}{1}$, where $n = (\_\_\_\_)$.	$(a \times b)$ or any counting number
72	The numbers $\frac{a}{b}$ and $\frac{b}{a}$ are multiplicative inverses (or reciprocals) since their product is the identity element for ______.	multiplication
73	The reciprocal of $\frac{a}{b}$ is the number $\frac{\_\_\_\_}{a}$.	b
74	The reciprocal of $\frac{3}{4}$ is ______.	$\frac{4}{3}$
75	The reciprocal of $\frac{7}{2}$ is ______.	$\frac{2}{7}$

76 The reciprocal of $\frac{3}{1}$ is ______. $\frac{1}{3}$

77 The reciprocal of $\frac{5}{1}$ is ______. $\frac{1}{5}$

78 The reciprocal of $\frac{0}{1}$:

☐ (a) is $\frac{0}{1}$

☐ (b) is $\frac{1}{0}$

☐ (c) does not exist.

78(a) Incorrect. $\frac{0}{1} \times \frac{0}{1} = \frac{0}{1}$ not $\frac{1}{1}$. See 78(c).

78(b) Incorrect. $\frac{1}{0}$ does not represent a rational number. See 78(c).

78(c) Correct. The number $\frac{0}{1}$ has no reciprocal since it is impossible to find a number such that the product of this number and $\frac{0}{1}$ is the identity for multiplication.

CHAPTER 22

DIVISION OF RATIONAL NUMBERS

We have defined specifically for rational numbers three of the four standard binary operations on numbers. In each case we have observed that rational numbers certainly involve different situations than whole numbers. That is, "addition" for whole numbers is by no means exactly the same as "addition" for rational numbers, nor is "multiplication" for whole numbers the same as "multiplication" for rational numbers. But the properties of the set of rational numbers under the operations of addition and multiplication are the same as for the set of whole numbers.

The definitions of the operations have been formulated in such a way that such standard properties as commutativity, associativity, distributivity, and special properties of the identity elements for addition and multiplication still apply.

The pattern we will follow in discussing "division" for rational numbers will, by now, be a familiar one. As before, we want to preserve, as far as possible, the special definitions and properties that apply to division of whole numbers. Multiplication will come into the matter, as you would expect. These considerations lead us to a definition of the operation of division of rational numbers.

22-1 Division of Rational Numbers

The definition of division of whole numbers is $a \div b = n$ if and only if $a = b \times n$. Thus, the definition of division of rational numbers is:

Definition: $\frac{a}{b} \div \frac{c}{d} = \frac{m}{n}$ if and only if $\frac{a}{b} = \frac{c}{d} \times \frac{m}{n}$, $b \neq 0$, $d \neq 0$, $n \neq 0$.

1 $\frac{2}{6} \div \frac{2}{3} = \frac{1}{2}$ since $\frac{2}{6} = \frac{2}{3} \times$ ______. $\frac{1}{2}$

2 $\frac{6}{10} \div \frac{3}{2} = \frac{2}{5}$ since $\frac{3}{2} \times \frac{2}{5} =$ ______. $\frac{6}{10}$

3 $\frac{2}{5} \div \frac{3}{4}$ is equal to:

☐ (a) $\frac{6}{20}$ ☐ (b) $\frac{16}{30}$ ☐ (c) No correct answer given.

3(a) Incorrect. You found the product of $\frac{2}{5}$ and $\frac{3}{4}$. Choose a different response.

3(b) Correct. $\frac{3}{4} \times \frac{16}{30} = \frac{48}{120} = \frac{2 \times 24}{5 \times 24} = \frac{2}{5}$. See 3(c).

3(c) Incorrect. 3(b) is the correct response, but it is rather difficult to arrive at $\frac{16}{30}$ as the quotient $\frac{2}{5} \div \frac{3}{4}$. The following theorem provides a convenient algorithm for finding a fraction which represents the quotient of two given rational numbers.

Theorem:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}, \text{ if } b \neq 0,\ c \neq 0 \text{ and } d \neq 0.$$

Proof:

4 $\frac{a}{b} = \frac{a}{b}$ since the two fractions ______ (are, are not) identical. are

5 $\frac{a}{b} = \frac{(c \times d)}{(c \times d)} \times \frac{a}{b}$ since $\frac{(c \times d)}{(c \times d)}$ represents the identity element for ______ of rational numbers. multiplication

6 $\frac{a}{b} = \frac{(c \times d)}{(d \times c)} \times \frac{a}{b}$ by the ______ property of multiplication of whole numbers. commutative

7 $\frac{a}{b} = (\frac{c}{d} \times \frac{d}{c}) \times \frac{a}{b}$ by the definition of ______ of rational numbers. multiplication

8 $\frac{a}{b} = \frac{c}{d} \times (\frac{d}{c} \times \frac{a}{b})$ by the ______ property of multiplication of rational numbers. associative

9	$\frac{a}{b} = \frac{c}{d} \times (\frac{a}{b} \times \frac{d}{c})$ by the ______ property of multiplication of rational numbers.	commutative
10	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ by the definition of division of ______ numbers.	rational
11	$\frac{2}{5} \div \frac{3}{4} = \frac{2}{5} \times \frac{4}{3} = \frac{\underline{\qquad}}{15}$.	8
12	$\frac{3}{7} \div \frac{2}{6} = \frac{3}{7} \times \underline{\qquad} = \frac{18}{14}$.	$\frac{6}{2}$
13	$\frac{5}{9} \div \frac{5}{9} = \frac{5}{9} \times \frac{9}{5} = \frac{\underline{\qquad}}{45}$.	45
14	$\frac{0}{4} \div \frac{7}{3} = \frac{0}{4} \times \frac{3}{7} = \frac{\underline{\qquad}}{28}$.	0

22-2 Properties of Division of Rational Numbers

15	5 ÷ 2 ________ (is, is not) a whole number.	is not
16	The whole numbers ________ (are, are not) closed under the operation of division.	are not
17	If $b \neq 0$, $c \neq 0$, $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times$ ______ as the result of a theorem.	$\frac{d}{c}$
18	$\frac{a}{b} \times \frac{d}{c}$ is a rational number, since multiplication of rational numbers has the ______ property.	closure
19	Since $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ and $\frac{a}{b} \times \frac{d}{c}$ is a rational number, $\frac{a}{b} \div \frac{c}{d}$ ________ (is, is not) a rational number.	is

20 Thus, $\frac{a}{b} \div \frac{c}{d}$ is always a rational number if $b \neq 0$, $c \neq 0$ and $d \neq 0$, and division of rational numbers has the ______ property.

closure

21 $\frac{3}{5} \div \frac{1}{2} = \frac{\underline{\quad}}{5}$.

6

22 $\frac{1}{2} \div \frac{3}{5} = \frac{\underline{\quad}}{6}$.

5

23 Thus, $\frac{3}{5} \div \frac{1}{2} \neq \frac{1}{2} \div \frac{3}{5}$ and division of rational numbers does not have the ______ property.

commutative

24 $(\frac{3}{5} \div \frac{1}{2}) \div \frac{3}{4} = \frac{6}{5} \div \frac{3}{4} = \frac{\underline{\quad}}{15}$.

24

25 $\frac{3}{5} \div (\frac{1}{2} \div \frac{3}{4}) = \frac{3}{5} \div \frac{4}{6} = \frac{\underline{\quad}}{20}$.

18

26 From Frames 24 and 25, one can say that division of rational numbers does not have the ______ property.

associative

Does division in the set of rational numbers distribute over addition from the right but not over addition from the left?

27 Consider $(\frac{3}{5} + \frac{2}{4}) \div \frac{1}{2}$.

(a) $\frac{3}{5} + \frac{2}{4} =$ ______.

$\frac{11}{10}$

(b) $(\frac{3}{5} + \frac{2}{4}) \div \frac{1}{2} = \frac{11}{10} \div \frac{1}{2}$

$= \frac{11}{10} \times \frac{2}{1} =$ ______.

$\frac{22}{10}$ or $\frac{11}{5}$

28	Also, if $(\frac{3}{5} + \frac{2}{4}) \div \frac{1}{2}$ is distributive, then (a) $(\frac{3}{5} + \frac{2}{4}) \div \frac{1}{2} = (\frac{3}{5} \div \frac{1}{2}) + (\_\_\_\_\_ \div \frac{1}{2})$ (b) $\frac{3}{5} \div \frac{1}{2} = \_\_\_\_\_$; and $\frac{2}{4} \div \frac{1}{2} = \_\_\_\_\_$. (c) $(\frac{3}{5} \div \frac{1}{4}) + (\frac{2}{4} \div \frac{1}{2}) = \underline{\_\_ + \_\_ = \_\_}$.	$\frac{2}{4}$ $\frac{6}{5}$ $\frac{4}{4}$ or 1 $\frac{6}{5} + 1 = \frac{11}{5}$
29	Hence, for this example, $(\frac{3}{5} + \frac{2}{4}) \div \frac{1}{2} = (\frac{3}{5} \div \frac{1}{2}) + (\frac{2}{4} \div \frac{1}{2}) = \frac{11}{5}$ and division is _____ over addition from the right.	distributive
30	$\frac{5}{6} \div (\frac{1}{2} + \frac{2}{3}) = \frac{5}{6} \div \frac{7}{6} = \frac{5}{6} \times \frac{6}{7} = \underline{\frac{\quad}{42}}$.	30
31	If division were distributive over addition from the left, then the result would be as follows: $(\frac{5}{6} \div \frac{1}{2}) + (\frac{5}{6} \div \frac{2}{3}) = \frac{10}{6} + \frac{15}{12}$ $= \frac{120 + 90}{72} = \underline{\frac{\quad}{72}}$.	210
32	Since $\frac{30}{42} \neq \frac{210}{72}$, Frames 28 and 29 show that in the set of rational numbers division does not distribute over addition from the _____ (left, right).	left

CHAPTER 23

DECIMALS

In the last few chapters we have considered the rational numbers named as fractions in the form $\frac{a}{b}$, with a a whole number and b a counting number, and we have discussed ways of computing with such numbers, chiefly by manipulation of their fractional forms.

Another common way of naming rational numbers, as you know, is by decimals, sometimes called decimal fractions. This chapter considers this way of naming rational numbers, the operations using these numerals, and the justification of "rules" that are commonly stated for doing such operations. Several issues raised by this way of writing numbers also are discussed.

Decimal fractions force themselves on the attention of youngsters very early because of their use in our monetary system. More important is the fact that the decimal notation is used in virtually all technical, scientific and business computing for both the English system and Metric system of measurements. And, as will be discussed in Chapter 30, decimals provide the only convenient means we have of dealing with certain numbers that cannot be named with a fraction in the form $\frac{a}{b}$.

For the moment, however, we will regard our decimals as naming numbers which could just as well be named by whole numbers, fractions, or mixed numbers. Most of our discussion will deal with "terminating decimals" and their fraction equivalents, for example, $.7 = \frac{7}{10}$, $.78 = \frac{78}{100}$, and so on. Near the end of the chapter we will discuss some "repeating decimals" and their fraction equivalents, for example, $\frac{1}{3} = .33333 \ldots$.

23-1. Introduction

1	In the decimal .0132, the 3 represents three _______ of one.	thousandths
2	In the decimal .517, the 5 represents five _______ of one.	tenths
3	In the decimal .27, the 2 represents two tenths or twenty _______.	hundredths

4	In the decimal .15, the 5 represents five hundredths or five tenths of one ______. If you answered all four of the preceding frames correctly, proceed to Frame 24.	tenth

The following is a representation of the whole number 3105 in expanded notation:

$$3105 = (3 \times 1000) + (1 \times 100) + (0 \times 10) + (5 \times 1)$$

Figure 23.1

5	In the whole number 3105, the 3 represents three ______ and is in the thousands place.	thousands
6	In the whole number 3105, the 0 represents ______ tens and is in the tens place.	zero
7	In the whole number 3105, the 5 represents five ______ and is in the last or ones place.	ones
8	In the whole number 3105, the 1 is in the ______ place.	hundreds

In our base ten place value system each digit represents a certain number according to its place. The central idea of the base ten place value notation is that the value of each place immediately to the left of a given place is ten times the value of the given place, and the value of a place immediately to the right of a given place is one tenth the value of the given place.

9	In the whole number 37, the 3 is in the tens place and represents ______ ones.	thirty or 3×10
10	In the whole number 105, the 5 is in the ones place and represents five ______ of ten.	tenths

11 In the whole number 312, the 1 represents one ten or one ______ of one hundred.

tenth

12 In the whole number 5203, the 2 represents two ______.

hundreds

13 In the whole number 5203, the 2 represents two tenths of one ______.

thousand

14 Thus, in any whole number, 1 in the hundreds place may be thought of as ______ ______ of one thousand.

one tenth

To make our place value system serve for naming rational numbers as well as whole numbers, we simply extend this idea of place value by saying that there are places to the right of the ones place and that the value attached to each will be, as in whole numbers, one tenth of the place immediately to its left.

15 The digit 4 is in the ones place in the following numbers:

☐ (a) 340 ☐ (b) 24.6 ☐ (c) 30.4 ☐ (d) 04.3

15(a) Incorrect. The 0 is in the ones place, while the 4 is in the tens place. See 15(b) and 15(d).

15(b) Correct. The period or dot, called a decimal point, is used to indicate that the place on its immediate left is the ones place.

15(c) Incorrect. The ones place is occupied by the 0 and the 4 is in the tenths place. See 15(b) and 15(d).

15(d) Correct. The 4 is in the ones place. Note that the 3 is in the place designated as the tenths place. See 15(b).

16	In the decimal 3.4, the 4 represents four tenths of the place on the left which is the ______ place.	ones
17	In the decimal 0.25, the 2 represents two ______.	tenths
18	The tenths place in the decimal 0.05 is occupied by the digit ______.	0
19	In the decimal .32, the 2 represents two tenths of one ______.	tenth
20	Since $\frac{2}{10} \times \frac{1}{10} = \frac{2}{100}$, two tenths of one tenth is two ______.	hundredths
21	Thus, in the decimal .32, the 2 represents two ______.	hundredths
22	Since $\frac{1}{100} \times \frac{1}{10} = \frac{1}{1000}$, one tenth of one hundredth will be one ______.	thousandth
23	Thus, the 3 in the decimal 0.123 represents ______ thousandths.	three

$$\frac{35}{100} = \frac{30}{100} + \frac{5}{100} = \frac{3}{10} + \frac{5}{100}$$

Figure 23.2

24	The number sentence in Figure 23.2 written in decimal notation would be .35 = .30 + .05 = .3 + ______.	.05
25	.52 = ______ + .02	.5 or .50
26	(.7 + .03) + .002 = ______	.732

27	The decimal .53 is read as fifty-three hundredths and .72 would be read as ______ hundredths.	seventy-two
28	The number one hundred six thousandths would be written as the decimal ______.	.106
29	Five hundredths would be written as the decimal ______.	.05
30	The decimal 2.01 is read as two and one hundredths. The decimal point is represented by the word ______ in reading the decimal.	and
31	Twenty-one and sixteen hundredths would be written as the decimal ______.	21.16
32	In the diagram below the letters under the dashes represent names of the place values in the decimal system of notation. Fill in the names of the place values which are not already labeled. ___ ___ ___ . ___ ___ ___ (a) (b) (c) (d) (e) (f)	
	(a) __________	hundreds
	(b) tens	
	(c) __________	ones
	(d) __________	tenths
	(e) hundredths	
	(f) __________	thousandths

Decimals can be expreseed in expanded notation in much the same way as whole numbers. For example, $1.32 = (1 \times 1) + (3 \times \frac{1}{10}) + (2 \times \frac{1}{100})$.

33 Write 10.3 in expanded notation.

10.3 = ______

$(1 \times 10) + (0 \times 1) + (3 \times \frac{1}{10})$

34 The decimal .014 written in expanded notation would be: ______.

$(0 \times \frac{1}{10}) + (1 \times \frac{1}{100}) + (4 \times \frac{1}{1000})$

35 $(8 \times 10) + (3 \times 1) \times (0 \times \frac{1}{10}) + (1 \times \frac{1}{100})$ is the expanded notation for the decimal ______.

83.01

36 Four hundred and four tenths could be written as

- ☐ (a) 40.4
- ☐ (b) 400.4
- ☐ (c) $(4 \times 100) + (0 \times 10) + (0 \times 1) + (4 \times \frac{1}{10})$
- ☐ (d) $(4 \times 10) + (0 \times 1) + (4 \times \frac{1}{10})$

36(a) Incorrect. Recall that the word "and" represents the decimal point in reading a decimal. See 36(b) and 36(c).

36(b) Correct. This is the decimal form for the number. See 36(c).

36(c) Correct. This is the expanded notation form for the number. See 36(b).

36(d) Incorrect. Recall that the word "and" represents the decimal point in reading a decimal. See 36(b) and 36(c).

37	The decimal .3 represents $\frac{3}{10}$ or $\frac{\_\_\_\_}{100}$.	30
38	The decimal .3 represents $\frac{3}{10}$ or $\frac{\_\_\_\_}{1000}$.	300
39	The decimal .3 is the same as three hundred thousandths written as the decimal ______.	.300
40	Thus, .3 ____________ (does, does not) equal .300.	does
41	In a given decimal, the affixing of zeros to the right of the last digit on the right ____________ (does, does not) change the value of the given decimal.	does not
42	The decimal .172000 ____________ (does, does not) equal .172.	does
43	Thus, the deletion of one or more successive zeros from the right end of a decimal ____________ (will, will not) affect its value.	will not

Since it is possible to affix or delete zeros at the right end of a decimal, it is possible to represent any pair of decimals so that they both have the same number of decimal places to the right of the decimal point.

44	The two decimals .213 and .17 expressed as thousandths would be .213 and ______ respectively.	.170
45	The two decimals .098 and 1.3 expressed as ten thousandths would be ______ and ______ respectively.	.0980; 1.3000

The fact that any pair of decimals can be expressed so that each has the same number of decimal places to the right of the decimal point may be used to compare the two decimals.

46 .32 and .27 are read as thirty-two hundredths and twenty-seven hundredths respectively. .32 > .27 since the two decimals are expressed in the same parts of the basic unit, namely hundredths, and for the number of parts,

32 ______ 27. >
(<, =, >)

47 .182 > .030 since the parts of the basic unit are ______ and 182 > 30. thousandths

48 To compare .12 and .138, the parts of the basic unit would probably be:

☐ (a) tenths
☐ (b) hundredths
☐ (c) thousandths
☐ (d) ten thousandths

48(a) Incorrect. .12 and .138 do not differ in the tenths place which gives no basis for comparison. See 48(c).

48(b) Incorrect. .12 and .138 do differ in the hundredths place which might afford a basis for comparison, but additional explanation would be required. See 48(c).

48(c) Correct. .12 = .120 and .120 < .138 since 120 < 138. The decimals are expressed in the same parts of the basic unit, namely thousandths.

48(d) Incorrect. .12 = .1200 and .138 = .1380 and 1380 > 1200. A comparison can be made using ten thousandths. However, it is not economical in that it involves affixing more zeros than are necessary. See 48(c).

49	Thus, .32 $\underset{(<, =, >)}{\underline{\qquad\qquad}}$ .315 since 320 > 315.	>
50	And .09 $\underset{(<, =, >)}{\underline{\qquad\qquad}}$ .101 since ____ < 101.	<; 90

Summary

To tell whether one of a pair of decimals is "less than," "equivalent to," or "greater than" the other, each is expressed in terms of the same part of the basic unit. Then, the order is determined by the number of parts expressed as whole numbers.

23-2. Operations Using Decimals

Each time we have introduced a new set of numbers, or a new way of writing familiar numbers, we have developed ways of dealing with equivalence, less than or greater than relations, and we have defined ways of doing standard operations. Equivalence and order for decimals have just been dealt with.

To begin a discussion of operations, let us remind ourselves that any such discussions should provide both conceptual models for the process at hand and efficient computational procedures. The conceptual aspect of the operations using terminating decimals can be very quickly disposed of by remarking that since they are only different ways of writing rational numbers, exactly the same models that were used for fractions suffice to give meaning to the operations with decimals. That is to say that for each such decimal used in an operation there is an exactly equivalent fraction of the form $\frac{a}{b}$, where b is some power of 10, so that using these equivalent fractions, the models and concepts previously discussed will apply. For that matter, any operation could be done merely by changing the decimals to fractions and using the computational procedures already discussed. It is convenient, however, to have ways of dealing directly with decimals via the usual operations.

We cannot dispose of these computational procedures very easily. Although we have fairly simple rules of thumb to tell us how to get answers, these rules are seldom well understood and are often incorrectly applied. This is especially true for the operation of division.

In the following sub-programs of this chapter we consider addition, subtraction, multiplication and division on decimals. In most cases the procedure amounts to first a computation with whole numbers, then some rule to place properly the decimal point in the answer.

51	$23.051 + 3.10721 =$ ______	26.15821
52	$.03142 - .001934 =$ ______	.029486
53	$12.1 \times .043 =$ ______	.5203
54	$.01024 \div .128 =$ ______	.08

55 $[.4 + (3 \times .02)] \div 2.3 =$

☐ (a) 2

☐ (b) .438 approximately

☐ (c) .2

55(a) Incorrect. $[.4 + (3 \times .02)] \div 2.3 =$
$(.4 + .06) \div 2.3 = .46 \div 2.3 = .2$.
Continue to Frame 56.

55(b) Incorrect. See 55(a).

55(c) Correct. If you answered Frames 51, 52, 53 and 54 correctly, go to Frame 122.
If you missed any of the preceding frames, proceed to Frame 56.

23-3. Addition of Decimals

56	$.32 + .15 = \frac{32}{100} + \frac{}{100}$	15
57	Then, $\frac{32}{100} + \frac{15}{100} = \frac{}{100}$	47

58	And, $\frac{47}{100}$ in decimal form is ______.	.47
59	Hence, .32 + .15 = ______.	.47
60	$.033 + .59 = \frac{33}{1000} + \frac{\quad}{1000}$	590
61	Then, $\frac{33}{1000} + \frac{590}{1000} = \frac{\quad}{1000}$.	623
62	And, $\frac{623}{1000}$ in decimal form is ______.	.623
63	Hence, .033 + .59 = ______.	.623
64	$(2.04 + .79) + .115 = \left(\frac{2040}{n} + \frac{790}{n}\right) + \frac{115}{n}$ for n = ______.	1000
65	Then, $\left(\frac{2040 + 790}{1000}\right) + \frac{115}{1000} = \frac{(2040 + 790) + \underline{\quad\quad}}{1000}$.	115
66	And, $\frac{2830 + 115}{1000} = \frac{\quad}{1000}$.	2945
67	Thus, (2.04 + .79) + .115 as a decimal is ______.	2.945
68	.035 + .117 as a fraction is $\frac{152}{n}$ if n = ______.	1000
69	Thus, .035 + .117 as a decimal is ______.	.152
70	.0072 + .015 as a fraction is ______.	$\frac{222}{10000}$
71	Thus, .0072 + .015 as a decimal is ______.	.0222
72	.23 expressed in a decimal representing thousandths would be ______.	.230
73	.3 expressed ina decimal representing thousandths would be ______.	.300

74 Then, the sum of .23 and .3 expressed in thousandths would be ______. .530

75 And, the sum of .23 and .3 expressed in hundredths would be ______. .53

76 (1.43 + .23) + .0079 =

☐ (a) 1.6679

☐ (b) 2.45

☐ (c) 1.66790

76(a) Correct. 76(c) also is correct.

76(b) Incorrect. In this example, the decimals 1.43 and .23 must be changed to agree in the number of decimal places with .0079.
For example:
(1.4300 + .2300) + .0079 = 1.6679.
Return to Frame 56 and continue therefrom.

76(c) Correct. 76(a) also is correct.

Summary

To add two or more decimals, annex zeros on the right of those decimals for which it is necessary so that all of them have the same number of decimal places to the right of the decimal point. Disregard the decimal points and proceed as in addition of whole numbers. Then place the decimal point so that the resulting sum has the same number of decimal places as each of the addends.

23-4. Subtraction of Decimals

77 $.35 - .15 = \frac{35}{100} - \frac{\_\_\_}{100}$ 15

78 Then, $\frac{35}{100} - \frac{15}{100} = \frac{\_\_\_}{100}$. 20

79	And, $\frac{20}{100}$ in decimal form is ______.	.20
80	Hence, .35 - .15 = ______.	.20
81	.39 - .033 = $\frac{390}{1000} - \frac{\underline{\qquad}}{1000}$.	33
82	Then, $\frac{390}{1000} - \frac{33}{1000} = \frac{\underline{\qquad}}{1000}$.	357
83	And, $\frac{357}{1000}$ in decimal form is ______.	.357
84	Hence, .390 - .033 = ______.	.357
85	(2.04 - .79) - .115 = $\left(\frac{2040}{n} - \frac{790}{n}\right) - \frac{115}{n}$ for n = ______.	1000
86	Then, $\frac{2040 - 790}{1000} - \frac{115}{1000} = \frac{(2040 - 790) - \underline{\qquad}}{1000}$.	115
87	And, $\frac{1250 - 115}{1000} = \frac{\underline{\qquad}}{1000}$.	1135
88	Thus, (2.04 - .79) - .115 as a decimal is ______.	1.135
89	.117 - .035 as a fraction is $\frac{82}{n}$ if n = ______.	1000
90	Thus, .117 - .035 as a decimal is ______.	.082
91	.015 - .0072 as a fraction is ______.	$\frac{78}{10000}$
92	Thus, .015 - .0072 as a decimal is ______.	.0078
93	.5 expressed in a decimal representing thousandths would be ______.	.500
94	.32 expressed in a decimal representing thousandths would be ______.	.320

95 Then, the difference .5 - .32 expressed in thousandths would be ______. .180

96 And, the difference .5 - .32 expressed in hundredths would be ______. .18

97 (1.43 - .23) - .0079 =

☐ (a) 1.1921

☐ (b) .41

☐ (c) 1.19210

97(a) Correct. 97(c) also is correct.

97(b) Incorrect. In the example, the decimals 1.43 and .23 must be changed to agree in number of decimal places with .0079.
For example:
(1.4300 - .2300) - .0079 = 1.1921 .
Return to Frame 77 and continue therefrom.

97(c) Correct. 97(a) also is correct.

Summary

To subtract two given decimals, annex zeros on the right of those decimals for which it is necessary so that all of them have the same number of decimal places to the right of the decimal point. Disregard the decimal points and proceed as in subtraction of whole numbers. Then place the decimal point so the resulting difference has the same number of decimal places as each of the given decimals.

To use subtraction with three or more decimals, the order of performing the subtractions must be indicated since rational numbers are not associative under the operation of subtraction.

23-5. Multiplication of Decimals

98	$.5 \times .15 = \frac{5}{10} \times \frac{\quad}{100}$	15
99	Then, $\frac{5}{10} \times \frac{15}{100} = \frac{\quad}{1000}$.	75
100	And, $\frac{75}{1000}$ in decimal form is ______.	.075
101	Hence, $.5 \times .15 =$ ______.	.075
102	$.03 \times .040 = \frac{3}{100} \times \frac{\quad}{1000}$	40
103	Then, $\frac{3}{100} \times \frac{40}{1000} = \frac{\quad}{100000}$.	120
104	And, $\frac{120}{100000}$ in decimal form is ______.	.00120
105	Hence, $.03 \times .040 =$ ______.	.00120
106	$(.2 \times .03) \times .25 = (\frac{2}{10} \times \frac{3}{100}) \times \frac{\quad}{100}$	25
107	Then, $(\frac{2}{10} \times \frac{3}{100}) \times \frac{25}{100} = \frac{6}{1000} \times \frac{25}{100} = \frac{\quad}{100000}$.	150
108	And, $\frac{150}{100000}$ in decimal form is ______.	.00150
109	Hence, $(.2 \times .03) \times .25 =$ ______.	.00150

110 (.25 × .03) × 1.0 =

☐ (a) 750.

☐ (b) 75.000

☐ (c) .00750

☐ (d) .0750

110(a) Incorrect. See 110(c) then return to Frame 98.

110(b) Incorrect. See 110(c) then return to Frame 98.

110(c) Correct. (25 × 3) × 10 = 750 and there are (2 + 2) + 1 decimal places in the product. Hence, (.25 × .03) × 1.0 = .00750.

110(d) Incorrect. See 110(c) then return to Frame 98.

Summary

To multiply two or more decimals, disregard the decimal point in each of the factors, then proceed as in multiplication of whole numbers. Determine the number of decimal places in each factor, find the sum of these numbers, then place the decimal point so that the product has this many decimal places.

If there are not enough digits in the product to accomodate the required number of decimal places, supply zeros between the decimal point and the left digit of the whole number product.

23-6. Division of Decimals

Definition: a ÷ b = n if and only if a = b × n where a, b and n belong to the same set of numbers and b is not the additive identity.

The above definition is generally the one used for division with a set of numbers. Just as it is used in division of whole numbers and rational numbers, it will be used in division of decimals.

111	.015 = .03 × ______	.5
112	Thus, .015 ÷ .03 = n if n = ______.	.5
113	.032 = .08 × ______	.4
114	Thus, .032 ÷ .08 = n if n = ______.	.4
115	.450 = .5 × ______	.90
116	Thus, .450 ÷ .5 = n if n = ______.	.90
117	.45 ÷ .5 = n if n = ______	.9
118	.18 ÷ 6 = n if n = ______	.03
119	1.0 ÷ 2 = n if n = ______	.5

120 1.5 ÷ 2 =

☐ (a) 7.5 ☐ (b) .75 ☐ (c) .075

120(a) Incorrect. 2 × 7.5 = 15.0 not 1.5. If you replace 1.5 with 1.50, then 2 × .75 = 1.50 and .75 is the correct response.

120(b) Correct. 1.5 ÷ 2 = 1.50 ÷ 2 = .75, since 2 × .75 = 1.50 = 1.5.

120(c) Incorrect. 2 × .075 = .150 not 1.5. If you replace 1.5 with 1.50, then 2 × .75 = 1.50 and .75 is the correct response.

121 1.50 ÷ .002 =

(a) 75 (b) 750 (c) .750

121(a) Incorrect. .002 × 75 = .150 not 1.50.
See 121(b).

121(b) Correct. 1.50 ÷ .002 = 1.500 ÷ .002 = 750,
since .002 × 750 = 1.500 = 1.50.

121(c) Incorrect. .002 × .750 = .001500 not 1.50.
See 121(b).

In the preceding frames the technique of finding the missing factor was used to obtain the quotient of two decimals. Since the same definition division is used for both whole numbers and decimals, whole number division may be used as a part of a different technique for division of decimals.

In this technique we disregard the decimal point and proceed as in whole number division.

122 Disregarding the decimal point, the division
$50\,\overline{)\,2.00}$ gives the whole number ______. 4

123 Disregarding the decimal point, the division
$.03\,\overline{)\,6}$ gives the whole number ______. 2

124 Disregarding the decimal point, the division
$50\,\overline{)\,.0300}$ gives the whole number ______. 6

After performing the whole number division, the next step is to place properly the decimal point in the whole number quotient to obtain the quotient of the two decimals.

In multiplication of a pair of decimals, the number of decimal places in the product is equal to the sum of the numbers of decimal places in the two factors.

The divisor and quotient correspond to the two factors in multiplication. Thus, the number of decimal places in the dividend must equal the sum of the number of decimal places in the divisor and the quotient.

125 For $.5\overline{)\,.020}$ the decimal quotient would be:

☐ (a) 4 ☐ (b) .4 ☐ (c) .04 ☐ (d) .004

125(a) Incorrect. The number of decimal places in this response is 0. The number of decimal places in the divisor .5 is 1, but $0 + 1 \neq 3$, the number of decimal places in the dividend .020.

125(b) Incorrect. The number of decimal places in this response is 1. The number of decimal places in the divisor .5 is 1, but $1 + 1 \neq 3$, the number of decimal places in the dividend .020.

125(c) Correct. The number of decimal places in this response is 2. The number of decimal places in the divisor .5 is 1, and $2 + 1 = 3$, the number of decimal places in the dividend .020.

125(d) Incorrect. The number of decimal places in this response is 3. The number of decimal places in the divisor .5 is 1, but $3 + 1 \neq 3$, the number of decimal places in the dividend .020.

Using the principle that the sum of the number of decimal places in the divisor and the quotient equals the number of decimal places in the dividend, place properly the decimal point in the quotient for each of the following:

126 $\begin{array}{r} 5 \\ 8\overline{)\,.040} \end{array}$ .005

127 $\begin{array}{r} 4 \\ 50\overline{)\,2.00} \end{array}$ .04

128 $\begin{array}{r} 5 \\ .04\,\overline{)\,.20} \end{array}$ 5.

129 $\begin{array}{r} 5 \\ .8\,\overline{)\,.040} \end{array}$ .05

Sometimes a decimal division arises in which whole number division is impossible without some adjustment in the dividend. For example, in .3 ÷ 6, 3 is not divisible by 6 in whole numbers. However, .3 = .30 and 30 is divisible by 6 in whole numbers. Thus, we adjust the dividend by affixing zeros on the right so that whole number division is possible.

130 Before the division $50\,\overline{)\,2}$ can be performed using whole numbers, the dividend 2 would probably be expressed as:

☐ (a) 2. ☐ (b) 2.0 ☐ (c) 2.00 ☐ (d) 2.000

130(a) Incorrect. In whole numbers 2 is not divisible by 50.

130(b) Incorrect. In whole numbers 20 is not divisible by 50.

130(c) Correct. In whole numbers 200 is divisible by 50.

130(d) Incorrect. In whole numbers although 2000 is divisible by 50 this is dealing with a larger number than necessary and probably would not be the one used. See 130(c).

131 Before whole number division can be used in $8\,\overline{)\,.04}$ the dividend .04 should be expressed as ______. .040

132 In the division $30\,\overline{)\,1.5}$ the dividend 1.5 should be expressed as ______ before whole number division can be used. 1.50

Other decimal division problems arise in which the number of decimal places in the dividend is less than the number of decimal places in the divisor. In these cases, it is impossible to find a whole number which when added to the number of decimal places in the divisor will give the number of decimal places in the dividend.

These cases require the same type of adjustment as those in which whole number division is not possible.

133 In the division $.0005\overline{\smash{)}\,.15}$ which of the following representations of the dividend should be used in order to place properly the decimal point in the quotient: (Read all four responses.)

☐ (a) .15 ☐ (b) .150 ☐ (c) .1500 ☐ (d) .15000

133(a) Incorrect. There are 4 decimal places in the divisor but only 2 decimal places in the response .15. Since there is no whole number which when added to 4 gives 2, an adjustment in the dividend is necessary.

133(b) Incorrect. There are 4 decimal places in the divisor but only 3 decimal places in the response .150. Since there is no whole number which when added to 4 gives 3, an adjustment in the dividend is necessary.

133(c) Correct. There are 4 decimal places in the divisor and 4 decimal places in the response .1500. And there is a whole number, namely 0, which when added to 4 gives 4, the number of decimal places in the dividend adjusted to .1500.

133(d) Correct. There are 4 decimal places in the divisor and 5 decimal places in the response .15000. And there is a whole number, namely 1, which when added to 4 gives 5, the number of decimal places in the dividend adjusted to .15000.

134 Place properly the decimal point in the quotient:

$$\begin{array}{r} 2 \\ .40\overline{)8.0} \end{array}$$

20.

135 Place properly the decimal point in the quotient:

$$\begin{array}{r} 3 \\ .075\overline{)2.25} \end{array}$$

30.

136 Place properly the decimal point in the quotient:

$$\begin{array}{r} 4 \\ .0012\overline{)4.8} \end{array}$$

4000.

137 The quotient for $.0025\overline{).1}$ is:

☐ (a) 4 ☐ (b) 40 ☐ (c) .4

137(a) Incorrect. You did not place properly the decimal point. Return to Frame 130 and the discussion preceding it; then continue therefrom.

137(b) Correct. Two zeros must be affixed to the dividend to affect whole number division. But at least three zeros must be affixed so that the number of decimal places in the dividend is at least 4, the number of decimal places in the divisor.

137(c) Incorrect. You did not place properly the decimal point. Return to Frame 130 and the discussion preceding it; then continue therefrom.

138 Find the quotient in $3.5 \overline{)\,.07}$.

☐ (a) .2 ☐ (b) 2 ☐ (c) .02

138(a) Incorrect. You did not adjust the dividend so that whole number division was possible. Return to the discussion preceding Frame 130 and continue therefrom.

138(b) Incorrect. You did adjust the dividend so that whole number division was possible, but you did not place properly the decimal point. Return to Frame 130 and the discussion preceding it.

138(c) Correct. Proceed to next frame.

139	11.27 ÷ .0023 = ______	4900.
140	.0595 ÷ 3.5 = ______	.017
141	496.8 ÷ .024 = ______	20700.
142	28.42 ÷ .2030 = ______	140.

From the chapters on division of whole numbers, recall that the number sentence n = (d × q) + r was used to express the division n ÷ d. Also, recall the techniques used in division of whole numbers.

	$14 \overline{)\,3642}$		
143	2800	______	200
	842		
144	______	50	700
	142		10
145	140		910
146	remainder = 2	______ (quotient)	

147 Thus, the above division written as a number sentence is 3642 = (14 × ______) + ______. 910; 2

We will use the same technique in division of decimals. We also will use the same method to place properly the decimal point in the quotient.

	23 / .3358		
148	______	.0100	.2300
	.1058		
149	.0920	______	.0040
	.0138		
150	______	.0005	.0115
	.0023		
151	.0023	______	.0001
152	remainder = 0	(quotient)	.0146

153 Thus, the above division written as a number sentence is .3358 = (23 × ____) + ____. — .0146; 0

154 In the following division, properly place all decimal points:

```
.31 / 15.593  |
      15 500  |  500
      00 093  |
      00 093  |    3
      --------+-----
           0  |  503
```

```
.31/15.593 |
    15.500 | 50.0
    00.093 |
    00.093 |   .3
    -------+-----
         0 | 50.3
```

155 Thus, the division written as a number sentence is 15.593 = (.31 × ____) + ____. — 50.3; 0

Often in physical situations we wish to determine a quotient having a prescribed number of significant digits.

In the following division, we want to find a quotient of 2 significant digits:

	4.2) .806		
156	____	.10	.420
	.386		
157	____	.05	.210
	.176		
158	____	.04	.168
159	remainder = ____	____ (quotient)	.008; .19

160 Thus, the above division written as a number sentence is $.806 = (4.2 \times \_\_\_\_) + \_\_\_\_$. — .19; .008

23-7. Changing Fraction Names to Decimal Names

Recall from Chapter 9 that a division represented in the form $n = (d \times q) + r$ is in best form provided $0 \le r < d$. Thus, in whole number division, the set of possible remainders for division by any non-zero whole number d is $\{0, 1, 2, 3, \ldots, (d - 1)\}$ if the division is in "best form."

161 The set of possible remainders for "best form" division by 3 would be {______}. — {0,1,2} or a set equal to this one.

162 $\{0, 1, 2, 3, \ldots, 41, 42\}$ would be the set of possible remainders for division by ______. — 43

163 The set of possible remainders for division by a given non-zero whole number is ______________ (finite, infinite). — finite

In whole number division, when the division is in "best form," the operation is completed. For example:

3) 7	
6	2
1	2

and $7 = (3 \times 2) + 1$.

However, by using decimals the same problem could be expressed

3. ⟌ 7.0	
6.0	2.0
1.0	
.9	.3
.1	2.3

and $7 = (3. \times 2.3) + .1$.

164

3 ⟌ 7		
____	2.	6.
1 = 1.0		
.9	.____	.3
.1 = .____		.10
.09	.03	
.____	2.33	.01

and 7 = (3 × ____) + .01 . 2.33

165	In Frame 164, the set of possible remainders is {0, 1, 2} or each of the remainders will correspond to the possible remainder ______.	1
166	If the division in Frame 164 is continued, the next remainder produced will correspond to the remainder ______.	1
167	The next digit in the quotient will correspond to ______.	3
168	If the division process is continued, the digits in the quotient will continue to be ______.	3's

169 The reason that the digits in the quotient repeat or are periodic is:

☐ (a) The number of possible remainders is finite.

☐ (b) A single possible remainder occurs more than once.

☐ (c) The quotient repeats because the remainder repeats.

All responses are correct and are basically equivalent.

170 Any rational number which has a repeating decimal expansion is ______. periodic

171 The decimal .3450 can be considered to be repeating by adding ______. zeros

172 If $.\overline{13} = .13131313 \ldots$, then $.213213213 \ldots =$ _____. $.\overline{213}$

173 Since in $.\overline{113}$ the parts repeat, the number is periodic and the period is:

☐ (a) one

☐ (b) two

☐ (c) three

173(a) Incorrect. The period is determined by the number of digits in the repeating portion.

173(b) Incorrect. See 173(a).

173(c) Correct. See 173(a).

174 The fraction $\frac{13}{99}$ represented as a repeating decimal is:

$$
\begin{array}{l|l}
99\,)\overline{\,13.0\,} & \\
\underline{\;\;9.9\;} & .1 \\
\;\;3.1 = 3.10 & \\
\qquad\;\; \underline{2.97} & .03 \\
\text{same possible remainder} \quad .13 = .130 & \\
\qquad\qquad\qquad\;\; \underline{.099} & \underline{.001} \\
\text{same possible remainder} \quad\quad .031 & \underline{\qquad}
\end{array}
$$

☐ (a) $.\overline{13}$ ☐ (b) $.\overline{1313}$ ☐ (c) $.\overline{131}$

174(a) Correct. When the 13 appears twice, period and periodicity are determined.

174(b) Correct. The 13 appears twice before the 31 appears twice and response 174(a) is preferred.

174(c) Incorrect. The next digit is a three.

We have seen how to find by division the decimal expansion of a given rational number. But, suppose we have the opposite situation, that is, we are given a periodic decimal. Does such a decimal represent a rational number? The answer is that it does, and we show this in the following paragraphs. The demonstration is a bit tricky, however, and involves some algebraic techniques that may not be familiar.

The problem may be approached by considering an example. Let us write the number 0.2424 ... and name it n so that $n = 0.\overline{24}$. The periodic block of digits is 24. If we multiply by 100, we obtain the relation:

$$100 \times n = 100 \times .242424 \ldots = 24.2424 \ldots$$

Then, since $100 \times n = 24.2424 \ldots$

and $n = 0.2424 \ldots,$

subtracting yields $99 \times n = 24$

so that $n = \frac{24}{99}$

or, in simplest form $n = \frac{8}{33}$.

We find by this process that $0.\overline{24} = \frac{8}{33}$.

The example above illustrates a general procedure developed by mathematicians for showing that *any periodic decimal represents a rational number*. We see, therefore, that *there is a one-to-one correspondence between the set of rational numbers and the set of periodic decimals*. It would be possible then for us to define the rational numbers as numbers represented by all such periodic decimals.

Now a question naturally arises about non-periodic decimals. What are they? Certainly not rational numbers. The fact that such non-periodic decimals exist should suggest that there are numbers which are not rational numbers. These non-rational numbers are called *irrational* and will be discussed in Chapter 30.

Computation with non-terminating decimals presents many problems, as can easily be verified by attempting to find the product

$(.333 \ldots \times .2727 \ldots)$.

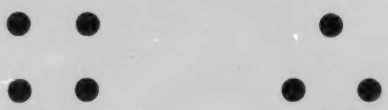
CHAPTER 24

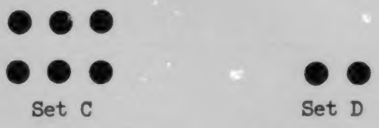
RATIO, RATE, PERCENT

24-1. Comparing Sets

In the study of whole numbers and of rational numbers, one usually considers a physical situation, first looking at its characteristic qualities and properties. From this look at the physical world, one tries to extract the ideas and properties of numbers which are basic to the study of mathematics.

The way in which certain sets of objects are alike was used to develop the concept of whole numbers. A set of 5 apples and a set of 5 letters of the alphabet can be put in a one-to-one correspondence. These sets have something in common. The fundamental property of interest is that of fiveness, denoted by the numeral 5. By considering joins of sets and arrays of sets, physical models for the ideas of addition and multiplication were exhibited.

1	Consider the following sets:  Set A ______ have more elements than Set B. (does, does not)	does
2	Compare the number of elements in Set A with the number of Set B in Frame 1. One may say that Set A has ______ elements and Set B has ______ elements.	4; 3
3	Also, compare the number of elements in the sets by saying Set A has ______ more element than Set B.	one or 1

4	Set A has been compared with Set B by ____________. (division, subtraction)	subtraction
5	Also, compare Set A with Set B by stating the relation as ______ elements to ______ elements.	4; 3
6	To compare Set A with Set 3 by division one could write ______ elements ÷ ______ elements. (In application, when these elements are of the same units, the name may be omitted.)	4; 3
7	Now compare Set B to Set A by division. _____ ÷ _____.	3 ÷ 4
8	When sets are compared by division the word ______ can be used for the symbol ÷.	to
9	 To compare Set C with Set D by subtraction one writes ______ - 2 = ______.	6 - 2 = 4
10	This means that Set C has ______ more elements than Set D.	4
11	To compare Set C with Set D by division, one writes ______ ÷ ______ = 3.	6 ÷ 2 = 3
12	This means Set C has ______ times as many elements as Set D.	3

13	Thus, one can compare the number of elements in Set C with the number of elements in Set D by either ______ or ______.	subtraction; division
14	Since the word "to" can be used for the division symbol, the number of elements in Set C may be compared with the number of elements in Set D by writing ______ to 2.	6
15	$3 \div 4$ means the same as ______ to ______, and either expression is called a ratio.	3 to 4
16	The ratio 6 to 5 can be written as $6 \div 5$ and the ratio 5 to 6 can be written as ______.	$5 \div 6$
17	$6 \div 5$ also may be written as a fraction $\frac{6}{5}$. $5 \div 6$ written as a fraction is ______.	$\frac{5}{6}$
18	Write $6 \div 5$ using the word "to." ______ ______ ______.	6 to 5
19	The colon : may be used instead of the word "to." Thus 6 : 5 can be written as ______ ______ ______.	6 to 5
20	Expressions using one and only one of the symbols $\div$, :, or the word "to" are called ______.	ratios
21	Match the equal ratios by placing the numeral in the blank corresponding to the letter of the indicated ratio. (a) 5 : 8 (1) $\frac{1}{3}$ (a) ______ (b) 3 to 1 (2) 5 to 8 (b) ______ (c) 1 to 3 (3) $\frac{3}{1}$ (c) ______	 (2) (3) (1)

24-2. Ratio and Proportion

Often ratios are used to describe the same basic property that an element of one set corresponds to a certain number of elements of a second set.

22	Let A represent a set of apples and let C represent a set of cents. If one apple costs 4 cents, then 2 apples will cost 8 cents. If the ratio of the number of cents in C to the number of apples in A represents the amount of money you pay for the number of apples in A, and further if the number in C is 4 and the number in A is 1, then the ratio of the number in C to the number in A is ______ and this corresponds to the cost of an apple.	4 to 1
23	2 elements of Set A would correspond to ______ elements of Set C.	8
24	Then, $4 : 1 = 8 :$ ______.	2
25	Or, this can be written in fractional form as $\frac{4}{1} =$ ______.	$\frac{8}{2}$
26	Recall from Chapter 19 that two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal if $a \times d = c \times b$. We conclude that $\frac{4}{1} = \frac{8}{2}$ if $4 \times 2 = 1 \times$ ______.	8
27	We can state then that if two ratios are ______ they will form a proportion.	equal

28 Which of the following ratios is equal to $\frac{3}{4}$?

☐ (a) $\frac{4}{3}$

☐ (c) $6 \div 9$

☐ (b) $\frac{9}{12}$

☐ (d) $3 \div 4$

28(a) Incorrect. $\frac{3}{4}$ is not equal to $\frac{4}{3}$ since $3 \times 3 \neq 4 \times 4$.

28(b) Correct. $\frac{3}{4} = \frac{9}{12}$ since $3 \times 12 = 4 \times 9$.

28(c) Incorrect. $\frac{3}{4} \neq \frac{6}{9}$ since $3 \times 9 \neq 4 \times 6$.

28(d) Correct. $\frac{3}{4} = 3 \div 4$ since $\frac{3}{4}$ and $3 \div 4$ mean the same thing. 28(b) also is correct.

29	Express $2 : 8 = 4 : 16$ in fractional form: ______ = ______.	$\frac{2}{8} = \frac{4}{16}$
30	In the proportion $\frac{2}{8} = \frac{4}{16}$, the product (2×16) representing $(a \times d)$ is ______.	32
31	In the proportion $\frac{2}{8} = \frac{4}{16}$, the product (8×4) representing $(b \times c)$ is ______.	32
32	Thus, $a \times d = b \times c$ since $32 = 32$, and we conclude that the ratios $\frac{2}{8}$ and $\frac{4}{16}$ are ______.	equal
33	The statement $\frac{2}{8} = \frac{4}{16}$ is called a ______.	proportion
34	Consider the proportion $\frac{a}{b} = \frac{c}{d}$. If any three of the four elements a, b, c, d are given, one can find the fourth element. If $\frac{3}{4} = \frac{c}{12}$, then $3 \times 12 =$ _____ $\times 4$.	c
35	Thus, _____ $= 4 \times c$.	36

36	And, ____ = c.	9
37	If 2 apples cost 9 cents, how many apples can be bought for 36 cents? Using fractions, this proportion can be written as: $\frac{9 \text{ cents}}{2 \text{ apples}} = \frac{\_\_\_\_}{n \text{ apples}}$.	36 cents
38	Then, $9 \times n = \_\_\_\_$.	2 × 36 or 72
39	And, $n = \_\_\_\_$.	8
40	If 20 people can be served with 8 pounds of beef, how much beef will be needed to serve 30 people? This can be written by writing equal ratios $\frac{20 \text{ people}}{8 \text{ pounds}} = \frac{\_\_\_\_}{n \text{ pounds}}$.	30 people
41	Then, $20 \times n = \_\_\_\_ \times 30$.	8
42	And, $20 \times n = \_\_\_\_$.	240
43	Thus, $n = \_\_\_\_$.	12

44 If a man can walk 3 miles in one hour, how long will it take him to walk 10 miles if he does not stop?

☐ (a) 3 hours

☐ (b) $\frac{3}{10}$ hours

☐ (c) $\frac{10}{3}$ hours

☐ (d) 30 hours

44(a) Incorrect. $\frac{3 \text{ mi.}}{1 \text{ hr.}} \neq \frac{10 \text{ mi.}}{3 \text{ hr.}}$ since $3 \times 3 \neq 1 \times 10$.

44(b) Incorrect. $\frac{3 \text{ mi.}}{1 \text{ hr.}} \neq \frac{10 \text{ mi.}}{\frac{3}{10} \text{ hr.}}$ since $3 \times \frac{3}{10} \neq 1 \times 10$.

44(c) Correct. $\frac{3 \text{ mi.}}{1 \text{ hr.}} = \frac{10 \text{ mi.}}{\frac{10}{3} \text{ hr.}}$ since $3 \times \frac{10}{3} = 1 \times 10$.

44(d) Incorrect. $\frac{3 \text{ mi.}}{1 \text{ hr.}} \neq \frac{10 \text{ mi.}}{30 \text{ hr.}}$ since $3 \times 30 \neq 1 \times 10$.

24-3. Percent

A special kind of ratio of rate is that of percent. Here 100 is always the basis for comparison. In fact, percent means "per hundred." Thus 25 percent means 25 per hundred. When written as a ratio, this would be 25 : 100 or $\frac{25}{100}$ or .25. It may again be written as $\frac{1}{4}$ or as $\frac{2}{8}$ or as any other fraction equivalent to $\frac{1}{4}$. From this we see that percent can be treated as a special type of ratio which can be converted to equivalent fractional forms or to equivalent forms.

Definition: In general, any number $\frac{a}{b}$ can be expressed as a percent by finding the number c such that $\frac{a}{b} = \frac{c}{100}$.

By studying this pattern, we see that if any two of the three numbers a, b, c are given, we can find the third.

45 6 is $n =$ ____ percent of 8 since $\frac{6}{8} = \frac{n}{100}$. 75

46 20 percent of 50 is $n =$ ____ since $\frac{n}{50} = \frac{20}{100}$. 10

47 40 is 10 percent of $n =$ ____ since $\frac{40}{n} = \frac{10}{100}$. 400

48 Given: $\frac{30}{100} = \frac{6}{20}$. Then 6 is ____ percent of 20. 30

49 Given: $\frac{11}{25} = \frac{44}{100}$. Then ____ is 44 percent of 25. 11

50 Given: $\frac{50}{40} = \frac{125}{100}$. Then 50 is 125 percent of ____. 40

CHAPTER 25

CONGRUENCES AND SIMILARITIES

25-1. Congruence

Congruence may be defined as follows:

Definition: Two geometric figures which have the same size and shape are said to be congruent.

This is not a technical definition of congruence, but it tells us what we need to know at this time.

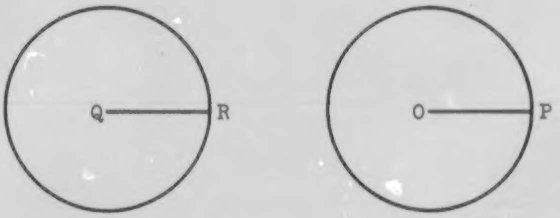
Congruence of line segments and congruence of angles were introduced in Chapter 15. In this chapter the notion of congruence of line segments and of angles is extended to congruence of triangles.

1 Congruence of segments and congruence of angles
________ be determined by direct comparisons — can
(can, cannot)
of their representations.

2 Congruence of any two plane figures ________ be — can
(can, cannot)
determined by direct comparisons of their representations.

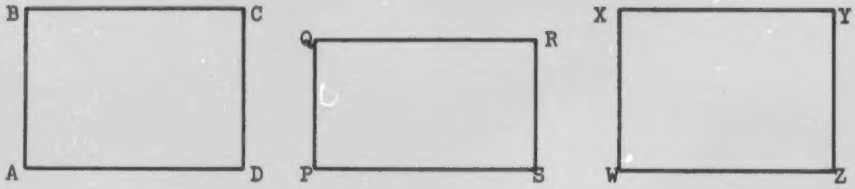
3 All radii of a given circle ________ congruent. — are
(are, are not)

4 In the two circles below, $\overline{OP} \cong \overline{QR}$.



Using a representation of circle Q, demonstrate that circle O is congruent to circle Q. From this and similar exercises one observes that two circles ________ congruent if their radii are — are
(are, are not)
congruent.

5 Consider the rectangles below:



Use the representation method to select the correct statements:

☐ (a) Two rectangles are congruent if their bases are congruent.

☐ (b) Two rectangles are congruent if their bases and angles are respectively congruent.

☐ (c) Two rectangles are congruent if their bases, angles, and heights are respectively congruent.

☐ (d) Two rectangles are congruent if their bases and heights are respectively congruent.

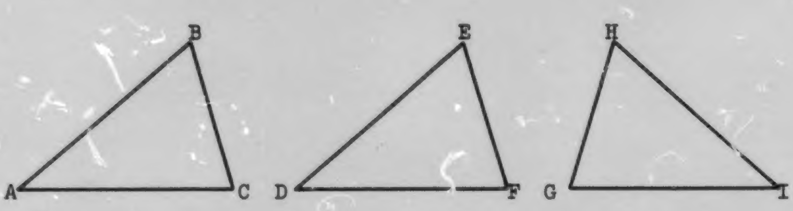
5(a) Incorrect. $\overline{AD} \cong \overline{PS}$, but rectangle ABCD is not congruent to rectangle PQRS.

5(b) Incorrect. $\overline{AD} \cong \overline{PS}$, and $\angle A \cong \angle P$, $\angle B \cong \angle Q$, $\angle C \cong \angle R$, $\angle D \cong \angle S$, but rectangle ABCD is not congruent to rectangle PQRS.

5(c) Correct. See also 5(d).

5(d) Correct. Notice that these two conditions are all that is necessary because all rectangles, by definition, have congruent (right) angles.

6 Given the following congruent triangles:



Using the representation method, select the correct response:

☐ (a) $\overline{AB} \cong \overline{DE}$; $\angle B \cong \angle E$;
$\overline{AC} \cong \overline{DF}$; $\angle A \cong \angle D$;
$\overline{CB} \cong \overline{FE}$; $\angle C \cong \angle F$.

☐ (b) $\overline{DE} \cong \overline{IH}$; $\angle D \cong \angle I$;
$\overline{DF} \cong \overline{IG}$; $\angle E \cong \angle H$;
$\overline{FE} \cong \overline{GH}$; $\angle F \cong \angle G$.

☐ (c) $\overline{AB} \cong \overline{GH}$; $\angle A \cong \angle G$;
$\overline{AC} \cong \overline{GI}$; $\angle B \cong \angle H$;
$\overline{CB} \cong \overline{IH}$; $\angle C \cong \angle I$.

☐ (d) $\Delta ABC \cong \Delta DEF$

6(a) Correct.

6(b) Correct.

6(c) Incorrect. $\overline{AB} \ncong \overline{GH}$, $\overline{CB} \ncong \overline{IH}$, $\angle A \ncong \angle G$, $\angle C \ncong \angle I$. Note that $\overline{AC} \cong \overline{GI}$, but $\overline{AC} \cong \overline{IG}$ is preferable, so that matching parts are mentioned in corresponding order.

6(d) Correct.

7 In Frame 6 the pairs of congruent sides and angles are called corresponding pairs of sides and ______ in $\triangle ABC$ and $\triangle DEF$.

angles

8 $\overline{AB}$ and $\overline{DE}$ are ______ sides or segments.

corresponding

9 $\angle C$ ______ to $\angle F$.

corresponds

Plausible Statement: If two triangles are congruent, then the three sides of one are congruent respectively to the three sides of the other and the three angles of one are congruent respectively to the three angles of the other.

10 If all six pairs of corresponding sides and angles of two triangles are congruent, then the triangles are ______.

congruent

11 If the three sides of one triangle are congruent to the three ______ sides of another triangle, then the triangles are congruent.

corresponding

12 If one knows only that the three angles of one triangle are congruent to the three corresponding angles of another triangle, he has no way of knowing whether or not the triangles are ______.

congruent

13 If two angles and the side which lies between them of one triangle are congruent to the ______ two angles and the side which lies between them of another triangle, then the two triangles are congruent.

corresponding

14 If two sides and the angle which lies between them of one triangle are congruent to the corresponding two sides and the ______ which lies between them of another triangle, then the two triangles are congruent.

angle

15 If one knows that two pairs of sides and one pair of angles of two triangles are congruent, then he has no way of knowing whether or not the triangles are ______. congruent

25-3. Similarity of Triangles

Similarity may be defined as follows:

Definition: Two geometric figures which have the same shape but not necessarily the same size are said to be similar.

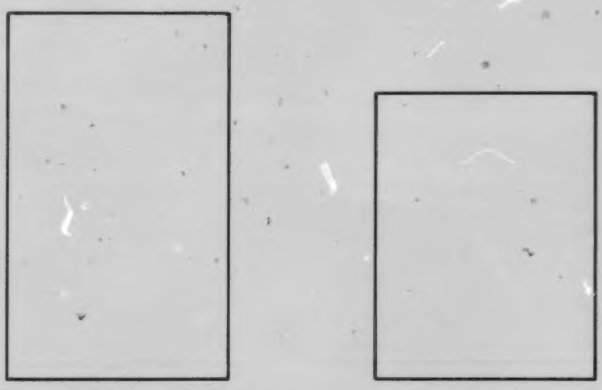
As in the definition of congruence, this is not a complete technical definition, but it tells us what we need to know at this time.

16 If three angles of one triangle are congruent respectively to three angles of another triangle, then the two triangles ______ (are, are not) necessarily congruent. are not

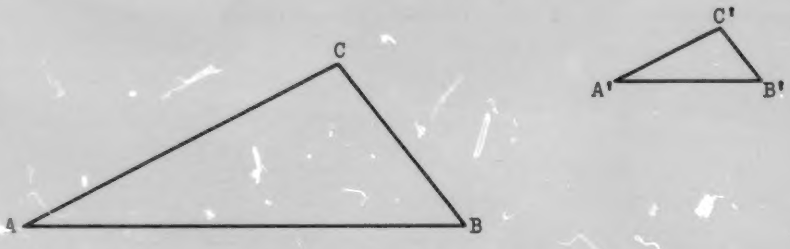
17 However, they do have the same shape even though they do not have the same ______. size

18 Two triangles which have three angles of one congruent to three ______ of the other are similar. angles

19 The statement in Frame 18 ______ (is, is not) true for the two rectangles below. is not



20 Given the two similar triangles below:



The measures (length) of their sides are:

m($\overline{AB}$) = 12
m($\overline{BC}$) = 6
m($\overline{AC}$) = 9
m($\overline{A'B'}$) = 4
m($\overline{B'C'}$) = 2
m($\overline{A'C'}$) = 3

Choose the correct responses:

☐ (a) m($\overline{AB}$) : m($\overline{A'B'}$) = 12 : 4 = 3 : 1

☐ (b) m($\overline{BC}$) : m($\overline{B'C'}$) = 6 : 2 = 3 : 1

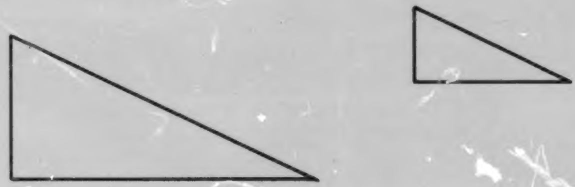
☐ (c) m($\overline{AC}$) : m($\overline{A'C'}$) = 9 : 3 = 3 : 1

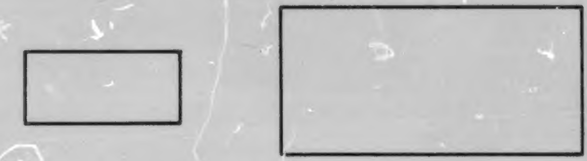
☐ (d) The measures of the corresponding sides of these two similar triangles have the same ratios.

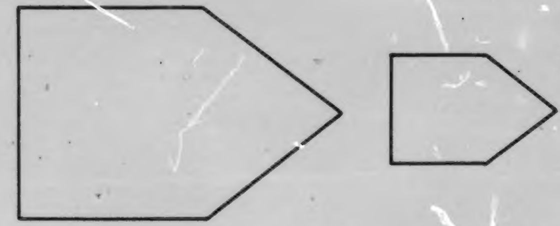
All responses are correct.

Mathematicians have found that the properties demonstrated in Frame 20 hold for any two similar triangles. Thus, if two triangles are similar, then the measures of their corresponding sides always have the same ratio.

21 Consider the following pairs of similar polygons:







In each pair the measures of the ______ sides have the same ratio. This generalizes for similar polygons. That is, if two polygons are similar, then the measures of their corresponding sides always have the same ratio.

corresponding

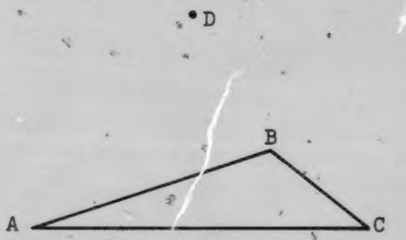
CHAPTER 26

SOLID FIGURES

In Chapter 14 simple closed curves and in particular triangles, rectangles, pentagons, and circles were introduced. The set of points inside a plane curve is the interior region of the curve. Simple closed curves of the plane are called two-dimensional figures. Figures which can not be contained in a plane are three-dimensional figures. Solid figures are three-dimensional. As with plane curves, the emphasis will be on the simple closed surface of the solid figures.

26-1. Pyramids

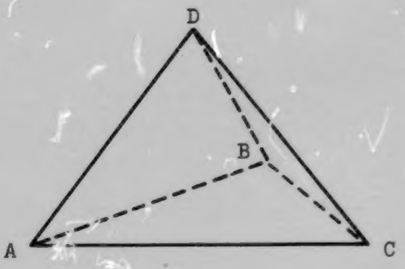
A pyramid is an example of a simple closed surface. It is made up of triangular regions and a polygonal region, the polygonal region forming the base.

1	To illustrate the boundary of the base of a pyramid, use a triangle which is a simple closed plane _______.	curve or polygon
2	In order to construct the three-dimensional figure called a pyramid, select a point not in the _______ of the base.	plane
3	Let triangle ABC be the base and D a point _______ in the plane of the triangle.  	not

One of the problems associated with solid figures or simple closed surfaces is that of drawing a picture in the plane to represent them. One learns to do this by carefully drawing and visualizing the results. The construction of three-dimensional models of geometric solids will be most helpful at this point.

4 Refer to the figure of Frame 3. Join the point D and the vertices of the triangle ABC forming segments _____, _____, and _____ called edges. See the figure below.

$\overline{AD}$; $\overline{BD}$; $\overline{CD}$



5 The sides of the base also are ______ of the pyramid.

edges

6 The point D is called a vertex of the pyramid. Point A also is a ______.

vertex

7 Two other vertices of the pyramid are ______ and ______.

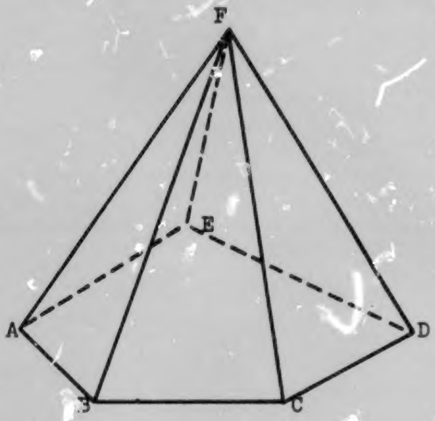
B
C

8 The triangular region bounded by the ______ ABD is called a lateral face of the pyramid.

triangle

9 Other lateral ______ are the triangular regions bounded by triangle BCD and triangle ACD.

faces

10	The ______ of a pyramid may be any polygonal region such as the pentagon ABCDE in the figure below. This pyramid is called a pentagonal pyramid.  	base
11	The lateral faces of a pyramid with a pentagonal base are the ______ regions and their boundaries sharing the vertex F.	triangular
12	The lateral faces of any pyramid are always ______ ______.	triangular regions
13	The base of a pyramid can be any plane ______ ______.	polygonal region
14	A pyramid is named by the polygon forming the base; hence, a pyramid with a triangular base is called a ______ pyramid.	triangular
15	A pyramid whose base is a quadrangle (quadrilateral) region is called a ______ pyramid.	quadrangular

16 The pyramid is a simple closed surface made up of the union of all points in the base and the ______ ______.

lateral faces

17 The interior of the simple closed surface is the set of points bounded by the ______ and the lateral faces.

base

18 The union of the set of points of a simple closed surface and the set of points in its ______ is called a solid region.

interior

The pyramid has been introduced as an illustration of a simple closed surface. A solid region consists of a simple closed surface and the set of points interior to the surface. It is common practice to use the word pyramid to mean the solid region. This, however, is not mathematically correct.

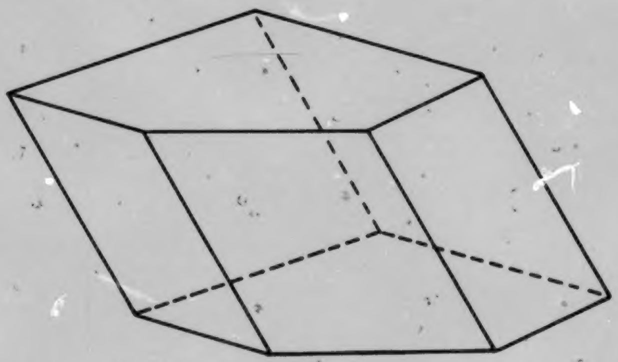
26-2. Prisms

19 The pyramid has one ______ ______ for its base.

polygonal region

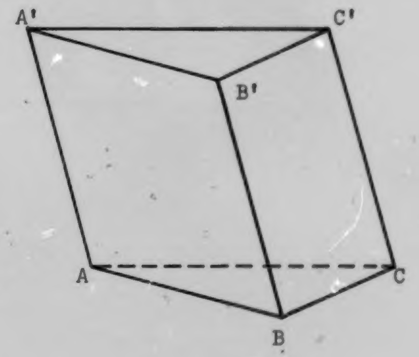
20 The prism, another simple closed surface, has two ______ both of which are congruent polygonal regions. See the figure below.

bases



21	In addition to the bases being ______ polygonal regions, their corresponding edges must be parallel.	congruent
22	A lateral face of a prism is the region bounded by the parallelogram formed by adjacent pairs of corresponding ______ on the bases.	vertices
23	The surface of an ordinary closed box is an example of a ______.	prism
24	The top and bottom of the box represent the ______.	bases
25	The sides of the box are the ______ ______.	lateral faces
26	The base of a prism is also a ______.	face
27	The intersection of two lateral faces is a ______ ______.	lateral edge
28	The edge formed by two lateral faces is a ______ ______.	lateral edge
29	All lateral edges of a prism are p______ to each other.	parallel
30	Since the lateral edges are parallel and the corresponding edges of the base also are ______, the lateral faces of a ______ are parallelogram regions.	parallel prism
31	If the lateral faces of a prism are all rectangular regions, the lateral edges form right angles with the bases and the prism is called a ______ prism.	right

Definition: The prism is a simple closed surface consisting of the sets of points bounded by two congruent polygonal regions (the bases which lie in parallel planes plus their boundaries) together with the sets of points bounded by parallelogram regions (the lateral faces) which are determined by the corresponding sides of the bases. The line segments joining corresponding vertices of the bases are the lateral edges.

32	As with pyramids, a prism is named by the kind of polygon forming the ______.	base
33	Consider the prism below. This prism is a ______ prism, since its bases are ______ ______. 	triangular; triangular regions
34	The bases of the triangular prism in Frame 33 are the triangular regions ______ and ______.	ABC; A'B'C'
35	The lateral edge containing the point B is ______.	$\overline{BB'}$
36	The lateral face containing points B and C is the ______ region BCC'B'.	parallelogram

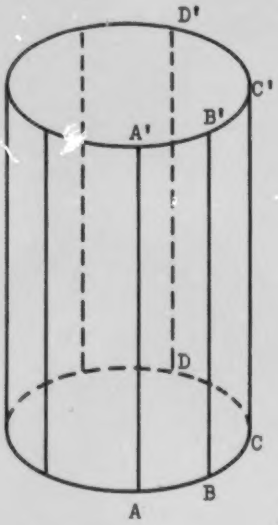
37	Another lateral face containing the point B is bounded by ______.	BB'A'A
38	The closed box, used as an example of a prism, is called a quadrangular prism since its bases are ______ regions.	quadrilateral
39	The special quadrangular prism for which each base and lateral face is a square region is called a ______.	cube
40	Three edges of a prism meet in a ______ called the vertex.	point
41	Each endpoint of an edge of a prism is a ______.	vertex
42	Each vertex of a prism is the endpoint of three ______.	edges

In the foregoing sub-program, the prism as a simple closed surface was considered. The concepts of interior and solid region were ignored, having the same definitions as those presented for pyramids.

26-3. Cylinders

A simple closed surface similar to the prism may be constructed by using two simple closed regions as the bases. Such a surface is called a cylinder.

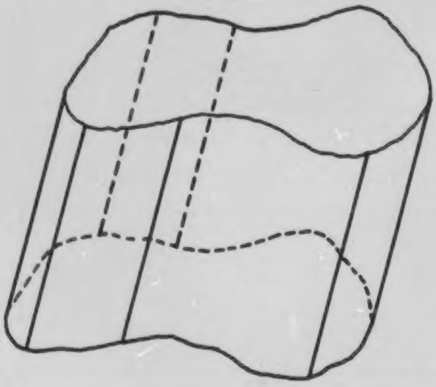
43 In the figure below, the ______ of the cylinder are circular regions. — bases



44 As with prisms, the bases of the cylinder must be congruent and must be in ______ planes. — parallel

45 The line segments $\overline{AA'}$, $\overline{BB'}$, et cetera, of the ______ in Frame 43 are called elements and join corresponding points in the bases. — cylinder

46 If the elements of a cylinder make right angles with the planes of the bases, the cylinder is called a ______ cylinder. — right

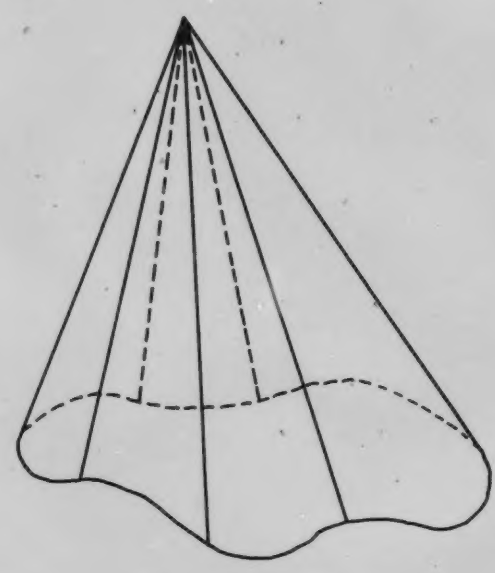
47	The figure below represents a cylinder since the bases are ______ closed regions in parallel planes. 	congruent
48	Not only are the bases parallel, but in any cylinder, the ______ are parallel to each other.	elements
49	The prism is a special case of a ______ since the bases are parallel and bounded by congruent simple closed curves (polygonal regions) and the sides can be considered as made of parallel segments.	cylinder
50	A juice can is a representation of a right ______ cylinder.	circular
51	A sardine can frequently is shaped so that it represents a ______ which is not circular.	cylinder

26-4. Cones

In the preceding section it was pointed out that the prism is a special case of a cylinder. In this section a generalization of the pyramid is considered.

52 A pyramid is determined by a polygonal region and a _______ not in the plane of the polygon. — point

53 A cone can be considered as the simple closed surface determined by a simple closed region and a _______ not in the plane of the cone. See the figure below. — point



54 The simple closed region determines the _______ of the cone. — base

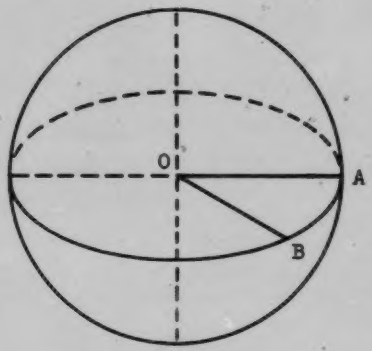
55 As with the cylinder, there may be no faces. However, line segments joining the point (a vertex) to the curve are the _______ of the cone. — elements

56	The point used for determining the cone is called the ______, as it was for pyramids.	vertex
57	The elements of a cone are line segments drawn from the ______ to points on the simple closed curve determining the base.	vertex
58	A cone with a circular base is called a ______ cone.	circular
59	The union of the sets of points making the ______ of the cone forms the lateral surface of the cone.	elements
60	If a cone is cut between the vertex and the base by a plane parallel to the base, the section made will be a ______ ______ ______.	simple closed curve

The foregoing two sections have been devoted to cylinders and cones. A cylinder is like a prism in that it has two congruent bases in parallel planes. The bases of cylinders may be determined by circles, but they can be determined by other simple closed curves. In a similar manner, pyramids and cones are compared.

26-5. Spheres

In this section another simple closed surface called the sphere is considered. It differs from those surfaces previously discussed in that it has no line segments on it.

61	A hollow rubber ball is a representation of the simple closed surface called a ______. See the model below.  	sphere
62	To every sphere there is an interior point called the ______.	center
63	If O is the center of a sphere, and A and B are any two points on the ______, then $\overline{OA} \cong \overline{OB}$.	sphere
64	Line segments from the ______ to the sphere are called radii of the sphere.	center
65	In a sphere, all ______ are congruent.	radii
66	A line segment joining two points on a sphere and passing through its center is called a d______ of the sphere.	diameter

67	Any ______ has the length of two radii.	diameter
68	A plane containing the center of a sphere intersects the sphere in a great circle. The radius of the great circle ______ (is, is not) congruent to the radius of the sphere.	is
69	The earth's surface almost represents a ______.	sphere
70	The equator of the earth represents a ______ ______.	great circle
71	The north and south poles are endpoints of a ______ of the earth.	diameter
72	A line of longitude on the earth is $\frac{1}{2}$ of a ______ ______ which passes through the north and south poles.	great circle
73	A line segment can meet (intersect) the sphere in at most ______ points.	two
74	The longest line segment separating all pairs of points on a sphere is the ______, and passes through the ______ of the sphere.	diameter center
75	The lines of latitude on the earth's surface are ______ formed by intersecting the sphere by a plane parallel to the plane of the equator.	circles
76	The circles giving the lines of latitude have radii ______ (greater than, congruent to, less than) the radii of a great circle and are called small circles.	less than
77	The radius of a great circle is ______ ______ the radius of a small circle.	greater than

26-7. Summary

In the foregoing program five different simple closed surfaces have been considered. There are many more. In mathematics, a simple closed surface is considered as the set of points on the boundary. Since the surface is closed, it has a well-defined interior. The union of the points on the simple closed surface and the points interior to it form a solid region.

CHAPTER 27

MEASURE OF AREAS

27-1. Introduction

The measure of areas involves plane regions. Recall that a plane region is the union of the set of points on a simple closed curve and the set of interior points. Examples are the plane regions of triangles, rectangles, polygons, circles, and simple closed curves in general. See Figure 27.1 below.

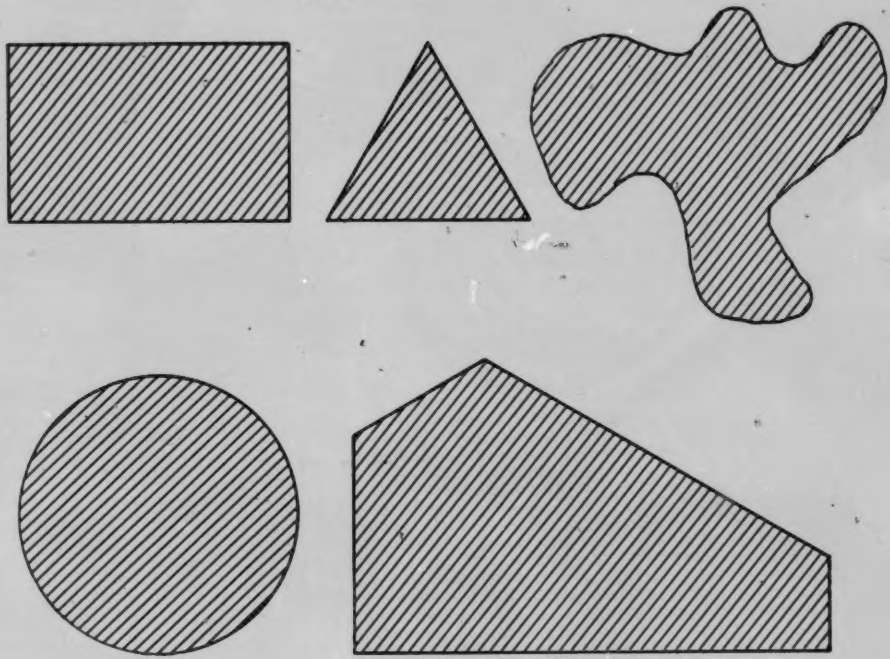


Figure 27.1

Although the words triangle, circle, parallelogram and the like properly refer to the set of points of the boundary, they are used in this chapter to represent the corresponding closed regions.

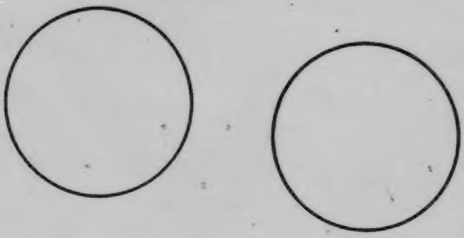
There are many features of measurement of areas analogous to measurement of line segments and angles as discussed in Chapter 16. For example, in comparing line segments one says that the first is less than, is congruent to, or is greater than the second. To obtain more precise results a unit represented by 1 is selected and applied to the line segment. By a process of counting one arrives at a measure of the line segment in terms of this unit.

The unit itself is arbitrary, but in practice certain standard units have been adopted for purposes of understandable communication.

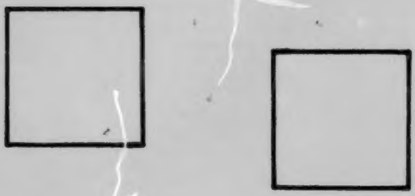
27-2. Comparisons of Regions

1 Given the pairs of figures below.

(a)



(b)



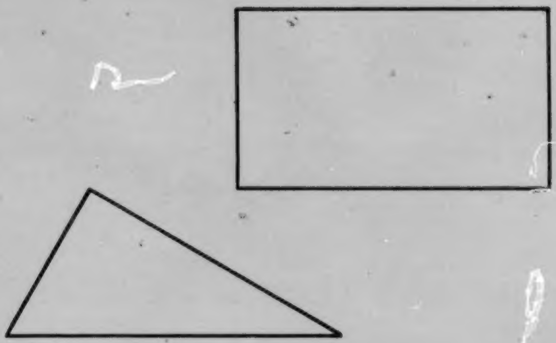
Compare by tracing one on transparent paper and placing the tracing over the other. The figures in each pair are ______.

congruent

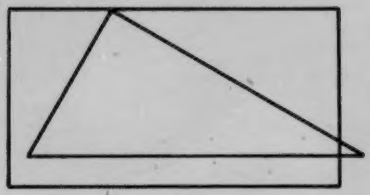
2 By the same procedure, compare one of the squares with one of the circles. The area of the circular region is ______ than the area of the square region.

greater

3 Given the following triangle and rectangle.

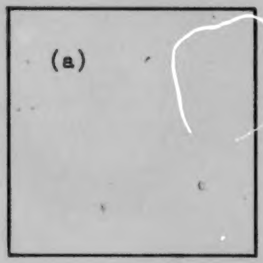


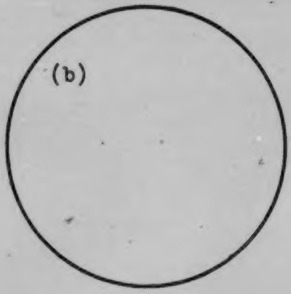
Place the rectangle on the triangle as indicated below.



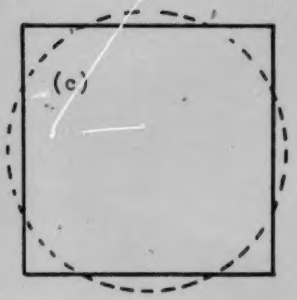
One concludes that the area of the rectangular region is ______________ the larger than
(larger than, less than, equal to)
area of the triangular region.

4 Consider the circle and the rectangle below.





Placing one of these figures on the other as indicated by the dotted line in (c), can one conclude that



☐ 4(a) (a) is greater than (b)?

☐ 4(b) (b) is greater than (a)?

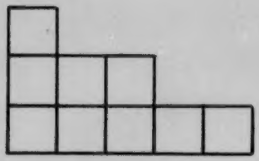
☐ 4(c) (a) is the same size as (b)?

The answer to all of these questions is NO since the area of the square and the area of the circle are approximately the same. Analogous situations occur whenever one compares areas bounded by unlike curved boundaries.

This sub-program has indicated that plane regions may be compared as to size or area in a manner analogous to the comparison of lengths of line segments. However, the comparison is more complicated in that it may be impossible to determine which area is the larger.

27-3. Units for Areas

In measuring line segments one selects an arbitrary unit of length and assigns to it a value of 1. One then attempts to "cover" the line segment by repeated applications of this unit without overlapping. In dealing with areas one follows the same procedure of selecting an arbitrary unit of area and attempting to "cover" the given area with a series of these units.

5	As the first step in measuring an area, select an arbitrary square region which is called the ______ of measure. 	unit
6	The number assigned to the chosen unit A in Frame 5 is ______.	1 or one
7	To measure the area of a given closed region, cover it with non-overlapping ______ squares.	unit
8	If the given region can be covered exactly as in the drawing below,  we find its area by ______ the units.	counting
9	We say that the area of the region in Frame 8 is ______ units.	9

10 We say that the measure of the region in Frame 8
is the number ______. 9

To facilitate the measurement of a plane region using an arbitrary unit of measure, one may construct a grid similar to the one in Figure 27.2. This grid serves a role similar to that of the ruler in measuring lengths and the protractor in measuring angles.

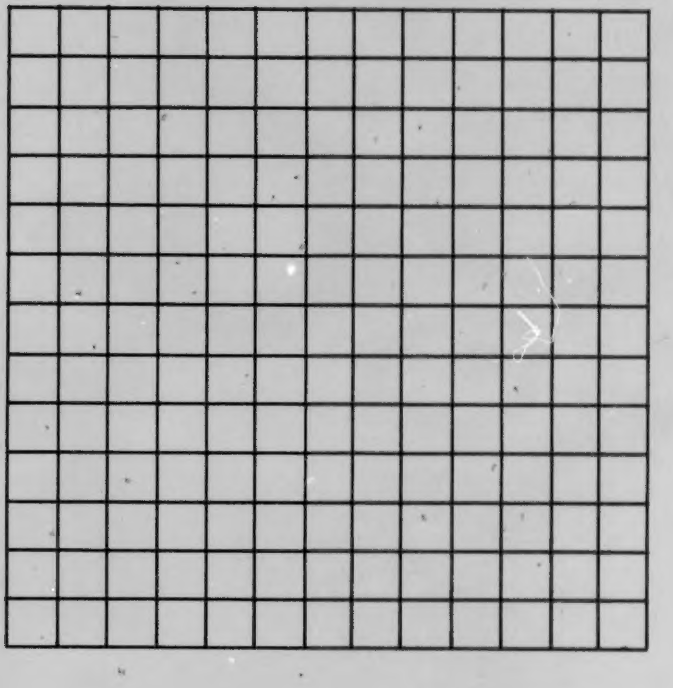
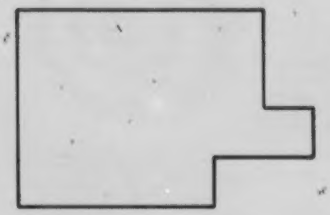


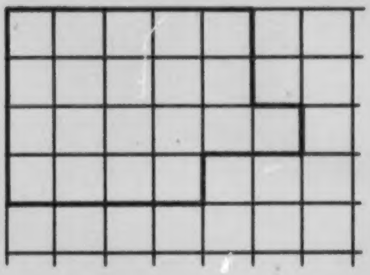
Figure 27.2

11 If a grid such as that in Figure 27.2 is fitted over the region below in (a), it will look like the figure in (b).

(a)



(b)



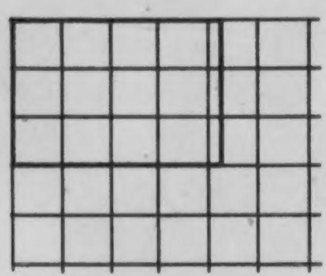
If we count the units of area, we find this to be ______ ______.

20 units

12 The measure of the plane region (a) in terms of the unit A is ______.

20

13 In applying the grid to another plane region, one may obtain results as in the figure below.



Counting the squares, the area is ______ ______ remembering that one counts whole number of units.

12 units (Could one say 13 units?

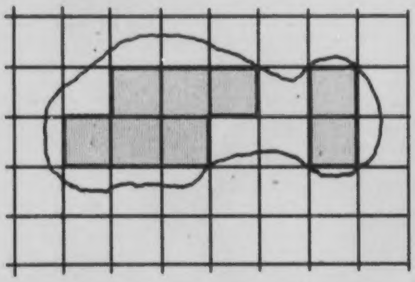
14 One certainly may say that the measure of the region in terms of the unit A is more than ______ and less than ______.

12;
15

15 If a more accurate answer is desired, it is necessary to select a smaller ______.

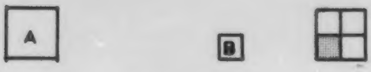
unit

16	In the same manner, other plane regions bounded by simple closed curves can be approximately measured. Consider the region below covered by the grid.	



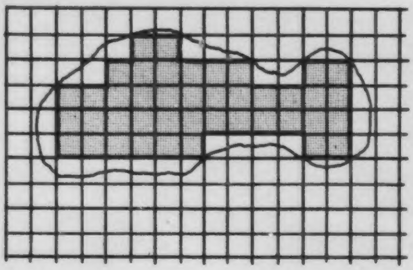
	Counting the square units shaded we say that the area of the region is at least ______ ______.	8 units
17	Counting also the squares partially covered, we may say the area is not more than ______ ______ in terms of the unit A.	26 units
18	As in measuring line segments, one increases the accuracy of measurement by ______ the size of the arbitrary unit A.	reducing or decreasing

19 Use a unit square with one-fourth the area, that is, a side one-half as long as the unit A.



$$B = \frac{1}{4} \text{ of } A$$

Cover the plane region of Frame 15 with this new unit. The drawing appears below covered by the new grid.



Counting the squares entirely within the simple closed curve shows that the measure of the area is at least ______. 42

20 By counting the squares covered or partially covered by the region, one finds the area to be not greater than ______ units. 75

21	In terms of the smaller unit B, the measure of the area of the region is greater than 42 and less than 75. However, by expressing the original unit as 4 of these smaller ones, our initial estimate was that the area was between ______ and ______.	4 × 8 = 32; 4 × 26 = 104
22	The ______ (smaller, larger) unit gives the better estimate of area of the region considered.	smaller

In this sub-program we have considered measurement of a plane region as a process of covering the region with non-overlapping units of area. The particular unit of area is arbitrarily chosen and usually taken to be a square.

As in measuring the length of a line segment, the basic process is counting; that is, one counts the units used to cover the plane region.

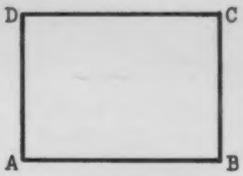
27-4. Formulas for the Area of a Rectangle

Instead of counting unit squares one seeks rules or formulas for finding areas of several common plane regions. These rules or formulas will be in terms of length of line segments which may be measured with a suitable unit of linear measure. The following sub-program will deal with rectangular regions.

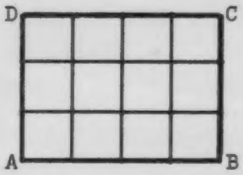
23	A rectangle is a 4-sided figure with ______ right angles.	4
24	The opposite sides of a rectangle are ______ and ______.	congruent; parallel

25 The plane region determined or bounded by a rectangle ABCD is the union of the set of points of the rectangle (or boundary) and the set of points of the ______ of the rectangle.

interior



26 Given the special rectangle ABCD in which $\overline{AD}$ measures exactly 3 units and $\overline{AB}$ measures exactly 4 units.



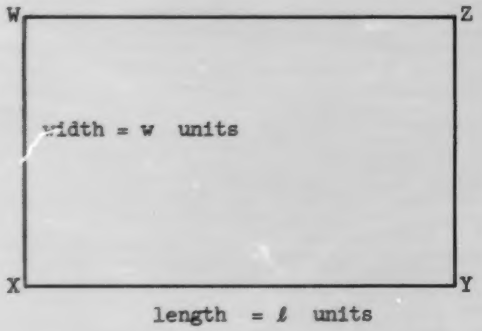
It is easy to see that unit squares with one linear unit on a side will cover this plane region with exactly ______ ______.

12 units

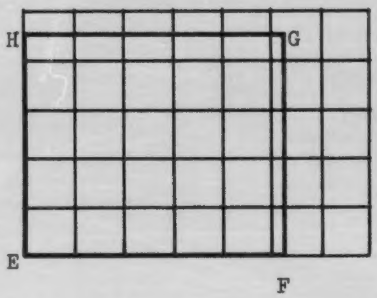
27 Dividing the rectangle into unit areas in this manner relates the measure of the area to the array of multiplication indicating the number sentence ______.

3 × 4 = 12

28	Given the rectangle PQRS such that its width measures exactly 6 inches and its length measures exactly 9 inches. This can be covered with squares one inch on a side. Counting them, one finds an area of ______ inch squares.	54
29	The same measure can also be obtained from the number sentence ______.	$9 \times 6 = 54$
30	If a rectangle WXYZ has a width exactly w units and a length exactly ℓ units, its area A can be found from the number sentence ______.	$A = w \times \ell$



31 Frequently the sides cannot be measured exactly with the unit used. In such cases one may estimate the area by superimposing a grid, as was done earlier in Frames 11 - 18. Such a grid is placed over the rectangle EFGH.



We may certainly say that the area is between
20 and _____ units. 30

32 One can estimate the measure of the area by using measurements of the line segments to the nearest unit. Measured to the nearest unit, the length of $\overline{EH}$ is 5 units and $\overline{EF}$ is 5 units. Using these approximate lengths, one obtains the approximate area of

$A \approx$ _____ × _____ $\approx$ _____ square units. 5; 5; 25

33	The accuracy of this measure may be increased by reducing the size of the covering units to $\frac{1}{4}$ units. The area will then be ______ 90 and 100.	between
34	In terms of the smaller unit, one-fourth the size of the original unit, the rectangle measures 9 units by 11 units to give an area of ______ of these quarter units.	99
35	Even though we speak of a quarter unit (quarter inch, et cetera), it is the ______, but is compared to some given standard, namely the original unit.	unit

When the line segments which form the rectangle can be measured exactly in terms of some unit, the region may be covered with a grid and the number of units of area counted; or the measure of the area, which is a real number, may be found by multiplying the length by the width. Hence, $A = \ell \times w$ where ℓ is the measure of the length and w is the measure of the width.

When the sides of a rectangle cannot be measured exactly in terms of the chosen unit, the area can be estimated by writing the length to the nearest unit or by selecting smaller units for both the lengths and the unit areas.

As a result of the preceding program we define area as follows:

Definition: The measure of the area of a rectangular plane region (or of a rectangle) is the number obtained as the product of the numbers measuring the length and the width or equivalently base and height. (The smaller the unit of length used, the more accurate the measure of the area will be.)

For brevity, we say "the area is the product of the base and the height" even though we should say "measure of" each time. We write the formula for the area of any rectangle as follows: $A = b \times h$ where A stands for the measure of the area, b the measure of the base, and h the measure of the height.

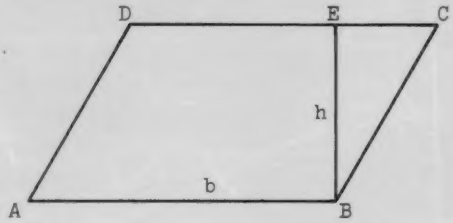
27-5. Areas of a Parallelogram and of a Triangle

To find the areas of parallelograms and triangles we do not usually use the method of covering the region with unit squares or a grid. By an analysis of the regions, we discover that their areas may be written in terms of rectangular plane regions whose areas can be found by the methods of the previous section.

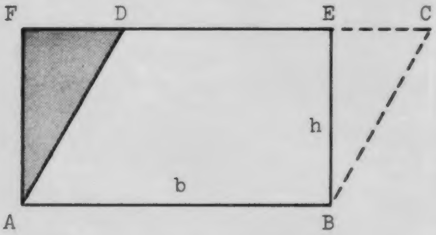
36 In a parallelogram the opposite sides are
congruent and ______. In general, the angles parallel
are not right angles.

37 The height of a parallelogram is the measure of
the ______ between opposite sides, measured distance
along a perpendicular.

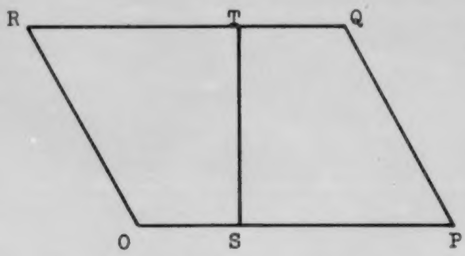
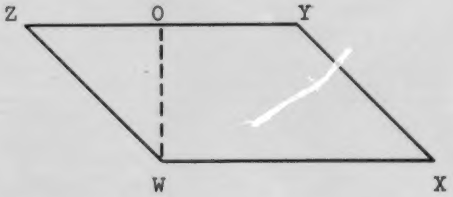
38 Consider the parallelogram ABCD whose base
$\overline{AB}$ measures b. The height h is measured
on the line segment $\overline{BE}$ drawn from B ______ perpendicular
to the two parallel sides $\overline{AB}$ and $\overline{DC}$.



39 Cut off the triangle BCE from the parallelogram
ABCE of Frame 38 and place it next to $\overline{AD}$ as
shown in the figure below.



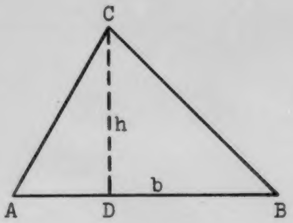
Then, the area of ABCD is equal to the area
of ______. ABEF

40	Since ABEF is a rectangle, its area is ______.	b × h
41	Hence the area of the parallelogram ABCD is equal to ______.	b × h
42	In any parallelogram OPQR, the height is the length of a line segment $\overline{ST}$ ______ to the base $\overline{OP}$.	perpendicular
		
43	Let the measure of $\overline{OP} = b$ and the measure of $\overline{ST} = h$. Then, the area of OPQR = ______.	b × h
44	In the given parallelogram WXYZ below, $m(\overline{WX}) = 4$ centimeters and $m(\overline{WO}) = 2$ centimeters.	
		
	The area is ______ square centimeters.	8

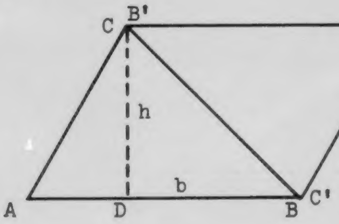
Definition: The measure of the area of a parallelogram is the number obtained as the product of the numbers measuring the base and the height. (The smaller the unit of length used, the more accurate the measure of the area will be.)

The formula for the area of any parallelogram can be written as: $A = b \times h$. Note that this is the same formula as the one used for the rectangle, but it must be clear that the distance h is the perpendicular distance between two parallel sides in both figures. However, for the rectangle, this is one of the sides, but for the parallelogram the height is shorter than the side.

45 Consider the triangle ABC with the base $\overline{AB} = b$ and the height $\overline{CD} = h$.



A model of the triangle is made and labeled A'B'C'. This model then is placed beside the triangle ABC so that $\overline{B'C'}$ coincides with $\overline{CB}$.



The resulting plane figure ABA'C is a ______.

parallelogram

46 The area of the parallelogram ABA'C is ______.

$b \times h$

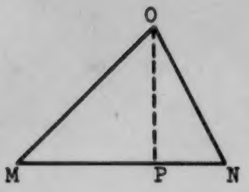
47 The area of the triangle ABC is ______ the area of the parallelogram ABA'C.

$\frac{1}{2}$ or one-half

48 Let A denote the area of a triangle, then the formula for the area of the triangle is ______.

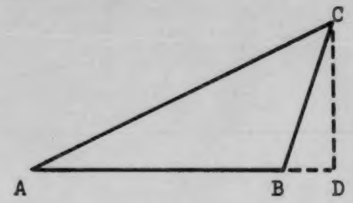
$A = \frac{1}{2} \times (b \times h)$

49 In the triangle MNO,
$m(\overline{MN}) = 3$ centimeters,
$m(\overline{OP}) = 2$ centimeters.
The area of triangle MNO in square centimeters is _______.



$\frac{1}{2}\times(2\times 3)$ or 3

50 In the triangle ABC,
$m(\overline{AB}) = 37$ millimeters,
$m(\overline{CD}) = 22$ millimeters.
The area is _______ square millimeters.



$\frac{814}{2}$ or 407

51 The unit of area measure used in Frame 50 is _______.

1 square millimeter

52 The measure of the area is the number of units in the area and hence the measure of the area of the triangle in Frame 50 is _______.

407

In this sub-program formulas for the areas of plane regions determined by parallelograms and triangles have been considered. This, together with the sub-program on rectangles, gives ways of finding the areas of many figures met in daily life.

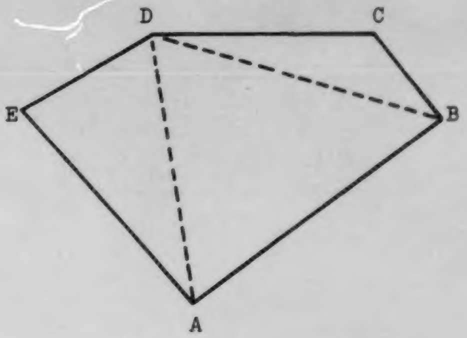
27-6. The Area of Polygons and Circles

A polygon is a region bounded by a series of line segments and may be divided into smaller regions consisting of triangles, rectangles and parallelograms. After this is done the smaller areas may be found and their sum will be the area of the original polygon.

53 A pentagon is a polygon with _____ sides.

5

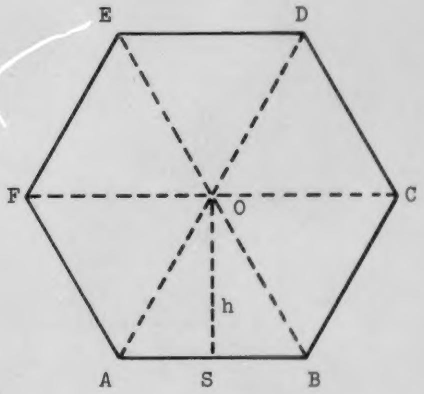
54 Join the vertex D and the vertices A and B by line segments dividing the pentagon into ______ as in the figure below. triangles



55 The area of the pentagon can be determined by ______ the areas of each of the triangles. adding

56 To determine the triangular areas, a base and corresponding height must be ______ on each triangle. measured

57 The six sided polygon ABCDEF below is called a regular hexagon.



The sides are all equal and if line segments $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are drawn, they will intersect in a point O. $\overline{AO} \cong \overline{BO}$ ____ $\overline{AB}$. $\cong$

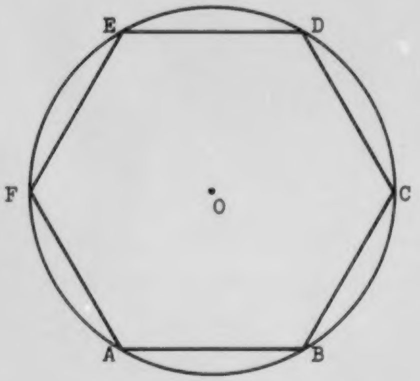
58 If the measure of $\overline{AB}$ is s and the height of triangle ABO is h, the area of the triangle ABO is ______.

$\frac{1}{2}\times(s\times h)$

59 Measurement shows that for each of the triangles the base and height are the same as for the triangle ABO. Hence, the total area of the hexagon is $6 \times$ ______ = ______.

$6\times\frac{1}{2}\times(s\times h) =$
$3\times s\times h$

60 Since the vertices A, B, C, D, E, F of the hexagon are all the same distance from O, a circle may be drawn around the hexagon as in the figure below.



The area of the circle is ______ (smaller, larger) than the area of the hexagon.

larger

61 The area of the circle in Frame 60 can be approximated by the area of the inscribed ______.

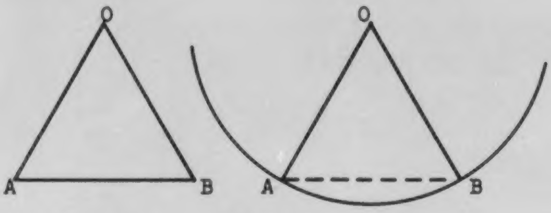
hexagon

62 The line segments drawn from O to the vertices of the regular hexagon divide the circle into ______ congruent regions called sectors.

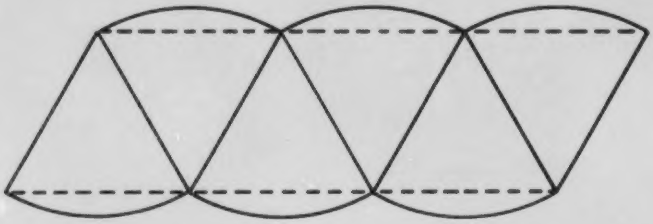
6 or six

63 The area of each sector is approximated by the area of the included ______ which has a smaller area.

triangle



64 The sectors can be arranged to look like

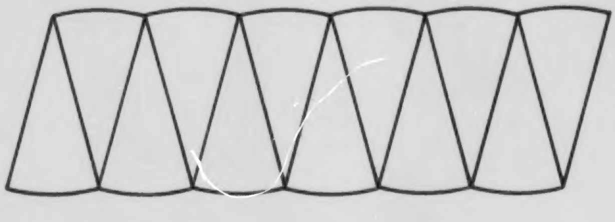


a figure with the ______ area as the circle.

same

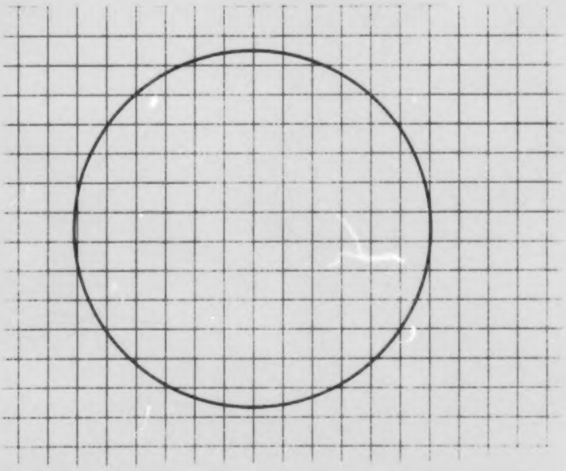
65 If each sector were cut in half to form two smaller congruent sectors and rearranged as in Frame 64, a figure looking more like a parallelogram with its height being the ______ of the circle and the length of its base approximately half the circumference of the circle. See the figure below.

radius



66	By halving sectors of Frame 65, a figure more nearly a parallelogram can be constructed having the ______ area as the circle.	same
67	The approximating parallelogram has a base with an approximate length of one-half the ______ of the circle and a height congruent to the radius.	circumference
68	Let r represent the radius, C the circumference and A the area. A formula for the area of a circle is ______.	$A = \frac{1}{2} \times (r \times C)$ or $(\frac{1}{2} \times C) \times r$

It is difficult to get more than an estimate of the area of a circle since one cannot draw by simple means a square or a rectangle which is equivalent in area to a given circle. In covering a circle with a grid, the difficulty is in counting the squares and making an estimate of the area of partial squares used to cover the circle.



This will give a rough estimate of the area of the circle, but is tedious and not very efficient.

If we use the formula written in Frame 68, namely $A = (\frac{1}{2} \times C) \times r$, the difficulty is in measuring C, the length of the circumference of the circle. This is almost impossible if the circle is on a sheet of paper or the chalkboard. If we use a tin can, a circular waste basket or the like, the circumference can be measured by the use of a string which may then be applied to a ruler or a grid. We may use this method in the classroom to find the area of a number of circles. We must measure the radius in terms of the same unit in each case.

69	The circumference of a coffee can measures, to the nearest $\frac{1}{4}$ inch, $15\frac{3}{4}$ inches or 63 quarter inches and the diameter measures 5 inches. The measure of the area of the bottom of the can in $\frac{1}{16}$ square inch units is A = ____ × ____ × ____ = ____.	$\frac{1}{2} \times 63 \times 10 = 315$
70	If a unit of $\frac{1}{10}$ centimeter is used, the measurement of the circumference of the same coffee can is 401 and the diameter is 128. The measurement of the area in terms of square millimeters ($\frac{1}{100}$ of a square centimeter) is A = ____ × ____ × ____ = ____.	$\frac{1}{2} \times 401 \times 64$ 12,832
71	For a Number 2 can, the circumference measures $9\frac{3}{4}$ inches and the diameter 3 inches. The area of the bottom of the can is ____ to the nearest $\frac{1}{16}$ inch square.	117($\frac{1}{16}$ inch squares)
72	The area of the bottom of the Number 2 can also can be expressed as $7\frac{5}{16}$ square inches to the nearest ____ ____ ____.	$\frac{1}{16}$ inch square
73	Refer back to Frame 69 in which we found 63 quarter inches to be the measure of the circumference of a coffee can and 20 quarter inches to be the measure of its diameter. The ratio of the circumference to the diameter, that is $\frac{C}{d}$ or C : d, is $\frac{63}{20}$ = ____.	3.15

74 The information of Frame 73 is sometimes written

$\frac{C}{d} = \frac{63}{20} = \frac{3.15}{1}$ or $C : d = 3.15 : 1$.

Using the measurements on the coffee can in terms of $\frac{1}{10}$ centimeter, we find

$\frac{C}{d} =$ ______ = ______.

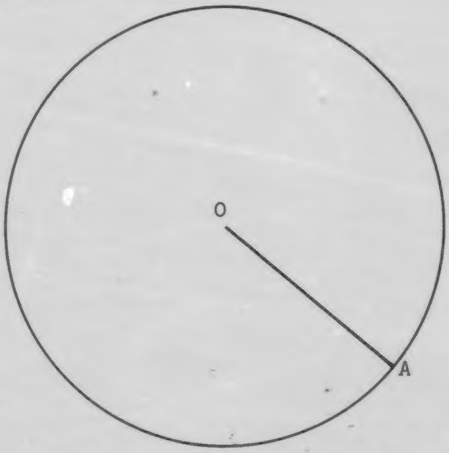
$\frac{401}{128} = \frac{3.13}{1}$

In Frames 73 - 74 two estimates for the ratio of the circumference to the diameter of a circle were found. If the two distances were measured more and more accurately, it could be found that the ratio would have values approximately equal to 3.14 : 1, 3.142 : 1, 3.1416 : 1, et cetera. Mathematicians know that this ratio is not a rational number. They know also that for any circle, the ratio of the measure of the circumference to the measure of the diameter will always be approximately 3.1416 no matter what units of measure are used. The number 3.1415 ... is called π (read "pi") and is related to the numbers C and d by the following equality:

$$\frac{C}{d} = \frac{\pi}{1}$$

This relation yields the formula for the circumference for any circle $C = \pi \times d$ and since $d = 2 \times r$, we also have $C = 2 \times \pi \times r$.

75 Given the circle with center O.



Using the quarter inch as a unit, it has a radius of ______ quarter inches.

5

76	Using the value 3.1416 for π, the circumference of the circle is $C = 2 \times \pi \times r =$ ______ quarter inches, which must be considered as 31 quarter inches to the nearest quarter inch.	31.416
77	The area of the circle in Frame 75 is given by the formula $A = \frac{1}{2} \times C \times r.$ Hence, $A =$ ______ or ______.	$\frac{1}{2}\times 31 \times 5 \left(\frac{1}{16}\text{ in. sq.}\right)$ $\frac{155}{2} = 77\frac{1}{2}$ or 78
78	Replacing C by $2 \times \pi \times r$ in the formula for the area of the circle gives the new formula $A = \frac{1}{2} \times (2 \times \pi \times r) \times r =$ ______.	$\pi \times r^2$

CHAPTER 28

MEASUREMENT OF SOLIDS

28-1. Introduction

The discussion of volumes of solid regions is more difficult than that of areas of plane regions primarily because of the difficulty in visualizing solid regions if pictures and diagrams of them are always in a plane.

1	As with the measure of area, one must select a ______ of volume.	unit
2	A cube is commonly selected as a unit of ______.	volume
3	In measuring a volume, one sees how ______ times the unit volume can be contained in the solid region, say n.	many
4	Similarly, one ______ the number of unit cubes necessary to enclose the solid region or volume being measured, say m.	counts
5	Designating the volume by V, one can say that the volume satisfies the number sentence ______.	$n \leq V \leq m$
6	If the volume to be measured can hold liquid, such as a jar or a barrel, then the volume can be determined to within a ______ of volume.	unit
7	To do this one takes a unit volume of liquid and pours it into the volume being measured. By ______ the number of unit volumes that can be held without overflowing one gets a number less than or equal to the volume.	counting
8	When the liquid overflows a ______ volume of liquid has been added than the container can hold.	larger

9 The volume usually lies ______ two successive whole number of units.

between

As with areas, volumes are frequently determined by computations instead of measurement, but as before an understanding of errors is needed.

28-2. Volume of a Rectangular Prism

For brevity, the words "volume of a rectangular prism" are used instead of the more accurate "volume of the solid region enclosed by a rectangular prism," and similarly for other solid figures.

10 A rectangular prism is a solid figure, each face of which is a ______.

rectangle

11 A cube is an example of a rectangular ______.

prism

12 Consider the model of the rectangular prism constituted by the following arrangement of unit cubes in Figure 28.1.

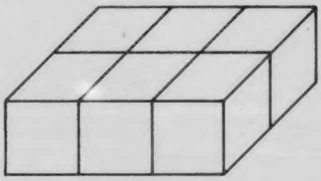


Figure 28.1

By counting the number of cubes (units), the volume is ______ units.

6

13 The measure of the volume also is given by the number sentence ______.

2 × 3 = 6

14 Take four rectangular prisms as in Frame 12 and stack them on top of each other as in Figure 28.2 below.

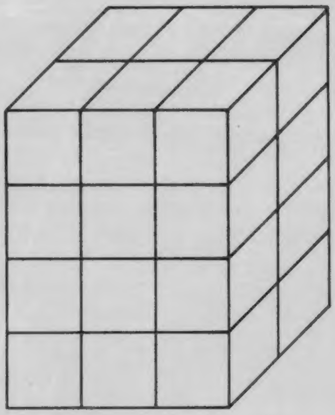


Figure 28.2

The volume of each layer is ______ units.

6

15 Since there are four layers, the volume of the rectangular prism in Frame 14 is ______ units.

24

16 Since the rectangular prism has length 3 units, width 2 units and height 4 units, the volume is expressed by the number sentence ______.

$3 \times 2 \times 4 = 24$

17 A rectangular prism with all edges congruent and each of unit length is a unit ______.

cube

18 As for the rectangular prism in Frame 14, when the volume was given by the number sentence $3 \times 2 \times 4 = 24$ the number sentence for the unit cube is ______.

$1 \times 1 \times 1 = 1$

19	Consider a rectangular prism with exact dimensions of length ℓ, width w, and height h. It appears that the number sentence describing the volume would be ______.	$\ell \times w \times h = V$
20	In the measure of area it was observed that the dimensions frequently were not exact; however, to estimate the area the lengths were measured to the ______ unit and the formula for the exact measure was used.	nearest
21	Previous experience in measuring area would indicate that the volume of a rectangular prism can be approximated by measuring the edges to the ______ unit and using the formula for exact measure.	nearest
22	A more accurate approximation can be found by using a smaller ______ of volume.	unit

Since the area of the base of a rectangular prism is equal to $\ell \times w$, we frequently say: The volume of a rectangular prism is the product of area of its base and its height.

28-3. Volumes of Certain Solids

23	It has been observed that the volume of a rectangular prism is the product of its height and the ______ of its base.	area

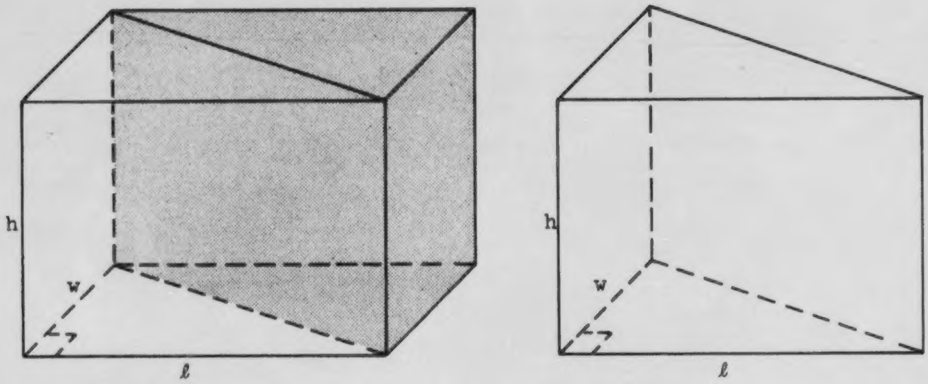


Figure 28.3

24	Consider the triangular prism formed by cutting the rectangular prism as indicated in Figure 28.3 above. Since the volume of the rectangular prism is $\ell \times w \times h$, the volume of the triangular prism is _____ $\times (\ell \times w \times h)$.	$\frac{1}{2}$
25	The area of the base of the triangular prism in Frame 24 is _______.	$\frac{1}{2} \times (\ell \times w)$
26	Frames 24 and 25 indicate that the volume of a triangular prism can be stated as _______ of the base times the height.	area

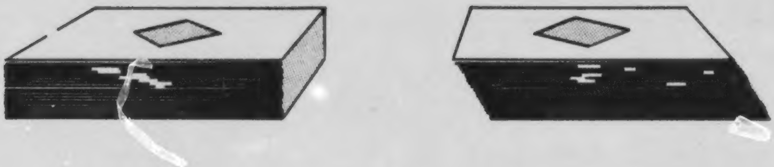


Figure 28.4

27	If a rectangular prism is not a right prism, a good physical model of the situation is a deck of cards which has been pushed into an oblique position as in Figure 28.4 above. Even in this case the volume of the prism is ______.	ℓ × w × h
28	For any prism, the volume can be found by multiplying the ______ of the base by the height.	area
29	A prism is a special case of a cylinder as indicated in Chapter 26. Hence, the volume of a cylinder is the ______ of the base times the height.	area
30	Similarly, the volume of cones and pyramids can be described by some relationship of the ______ of the base times the height.	area

An experiment gives the formula for finding the volume of pyramids and cones. Make a model of a given pyramid, then take a prism with base and height congruent to those of the pyramid. We find that if the pyramid is filled with sand and the sand is poured into the prism, the prism will be filled after three such pourings. The same is true for a cone and its corresponding cylinder.

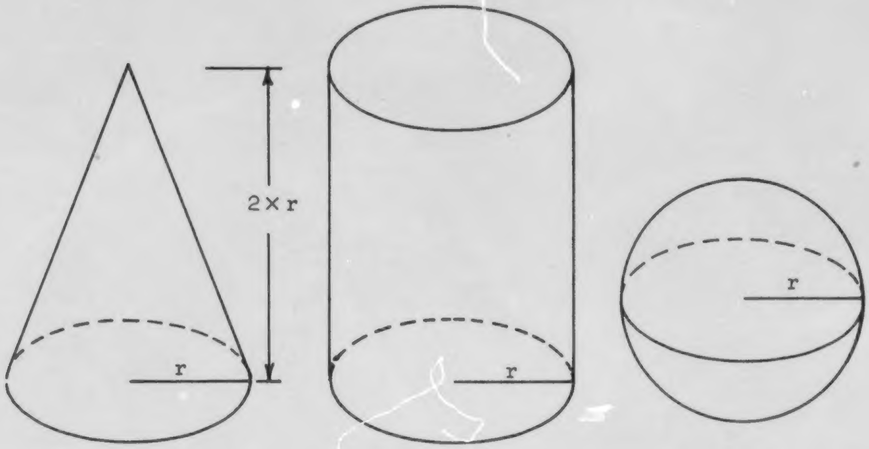
31 For any pyramid or cone,

$V = \frac{1}{3} \times$ ____ $\times h$.

B or area of base

28-4. Volume of a Sphere

In order to discover the volume of a sphere, consider the three solids in Figure 28.5 below.



right circular cone right circular cylinder sphere

Figure 28.5

32 The area of the base of the right circular cylinder is $B = \pi \times$ ____.

r^2 or $r \times r$

33 The volume of the right circular cylinder is $V =$ ____ $\times 2 \times r$.

$\pi \times r^2$

34 The base of the right circular cone has the same area as the base of the right circular cylinder, and the volume of the right circular cone is $V =$ ____ $\times \pi \times r^2 \times 2 \times r$.

$\frac{1}{3}$

It can be demonstrated by experimentation that the volume of the cylinder less the volume of the cone is equal to the volume of the sphere.

35 The volume of the sphere is the volume of the cylinder less the volume of the cone, that is,

$$\begin{aligned} V_{\text{of sphere}} &= 2 \times \pi \times r^3 - \frac{2}{3} \times \pi \times r^3 \\ &= \left(\frac{6}{3} - \frac{2}{3}\right) \times \pi \times r^3 \quad \text{(by the distributive property)} \\ &= \underline{\frac{4}{3} \times \_\_\_\_ \times \_\_\_\_} \quad \text{(the formula for the volume of a sphere).} \end{aligned}$$

$\frac{4}{3} \times \pi \times r^3$

28-5. Areas of Simple Closed Surfaces

36 The surface area of a cylinder, a cone, a pyramid, or a prism is found by determining the ______ of the base and the lateral surface.

area

37 The lateral surfaces of pyramids are triangular regions and the lateral surfaces of prisms are ______; hence, these areas can be found easily.

parallelograms

38 To determine the lateral area of a right circular cylinder, one can see in Figure 28.6 below that the area is $(2 \times B)$ + ______.

$(C \times h)$ or circumference times height

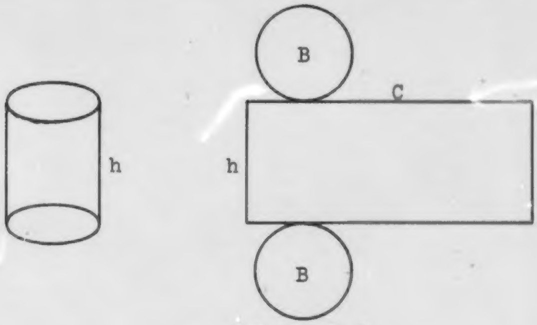


Figure 28.6

To determine the surface area of a cone, the cone can be flattened out on a plane as in Figure 28.7 below.

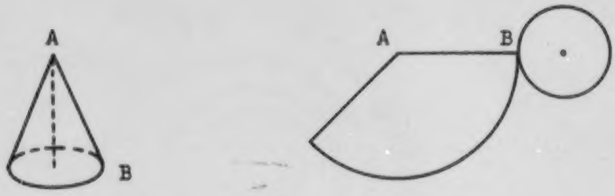


Figure 28.7

However, a formula in terms of the area of the circular base and the height is too complicated to be considered at this point.

The surface area of a sphere is more difficult to obtain since it is not possible to flatten out the surface of a sphere into a plane. However, a formula can be given:

$$\begin{aligned}\text{Surface area of a sphere} &= C \times 2 \times r \\ &= (2 \times \pi \times r) \times (2 \times r) \\ &= 4 \times \pi \times r \times r \\ &= 4 \times \pi \times r^2\end{aligned}$$

where $\underline{r}$ is the radius of the sphere and $\underline{C}$ is the circumference of a circle of the sphere whose radius is $\underline{r}$.

CHAPTER 29

NEGATIVE RATIONAL NUMBERS

Up to now, three different sets of numbers have been studied. They are:

(1) the counting numbers: 1, 2, 3, 4, 5, 6, 7, ... ;

(2) the whole numbers: 0, 1, 2, 3, 4, 5, 6, ... ;

(3) the rational numbers: 0, ..., $\frac{1}{2}$, ..., $\frac{2}{3}$, ..., $\frac{29}{7}$, ...

In this chapter we extend once again the concept of number and include the notion of direction. This will result in a new set of numbers: the set of positive, negative and zero rational numbers. From now on this new set of numbers will be called the set of *all rational numbers*, and the third set above will be called the *non-negative rationals*.

29-1. Membership in the Set of Rational Numbers

The definitions and subsequent discussions of our new set of numbers depends on the notion of an *ordered pair* of numbers.

1	The ordered pair (b, c) has ______ (how many) components, namely b and c.	two
2	In the ordered pair (b, c), b is the *first* component and c is the ______ component.	second
3	In an ordered pair, the components are separated by a comma and are enclosed in ______.	parentheses
4	The first component is always written on the ______ (right, left) and the second component is always written on the ______ (right, left).	left right

5 If 5 is the first component and 2 is the second component of an ordered pair, the ordered pair would be written as

☐(a) (2, 5) ☐(c) 5, 2

☐(b) (5 2) ☐(d) (5, 2)

5(a) Incorrect. The components are not in the given order.

5(b) Incorrect. The components are not separated by a comma. Thus (5 2) is not an ordered pair.

5(c) Incorrect. The components are not enclosed in parentheses. Thus, 5, 2 is not an ordered pair.

5(d) Correct. Note why each of the three other responses are incorrect.

6 The ordered pair (2, 3) is the same as

☐(a) (3, 2) ☐(b) (2, 3) ☐(c) (2, 4)

6(a) Incorrect. The pairs of components are the same but the order is different.

6(b) Correct. The pairs of components are the same and the order is the same.

6(c) Incorrect. The pairs of components are not the same.

Definition: Let a and b denote non-negative rational numbers. Then, the ordered pair (a, b) denotes a rational number and is defined as follows:

$$(a, b) = a - b = {}^{+}(a - b) \text{ if } a > b;$$
$$(a, b) = a - b = 0 \text{ if } a = b;$$
$$(a, b) = a - b = {}^{-}(b - a) \text{ if } a < b.$$

7	The ordered pair $(5, 2)$ = ____ or ____ .	$^{+}(5-2)$ or $^{+}3$
8	The ordered pair $(2, 5)$ = ____ or ____ .	$^{-}(5-2)$ or $^{-}3$
9	The ordered pair $(2, 7)$ = ____ or ____ .	$^{-}(7-2)$ or $^{-}5$
10	The ordered pair $(\frac{7}{5}, \frac{7}{5})$ = ____ or ____ .	$\frac{7}{5}-\frac{7}{5}$ or 0
11	The ordered pair $(\frac{7}{8}, \frac{4}{8})$ = ____ or ____ .	$^{+}(\frac{7}{8}-\frac{4}{8})$ or $^{+}\frac{3}{8}$
12	The ordered pair $(\frac{1}{6}, \frac{4}{6})$ = ____ or ____ .	$^{-}(\frac{4}{6}-\frac{1}{6})$ or $^{-}\frac{3}{6}$ or $^{-}\frac{1}{2}$
13	The ordered pair $(\frac{1}{3}, \frac{1}{2})$ = ____ or ____ .	$^{-}(\frac{1}{2}-\frac{1}{3})$ or $^{-}\frac{1}{6}$
14	The ordered pair $(\frac{1}{2}, \frac{1}{3})$ = ____ or ____ .	$^{+}(\frac{1}{2}-\frac{1}{3})$ or $^{+}\frac{1}{6}$

In previous chapters we used several different physical models for whole numbers and non-negative rational numbers to help illuminate different mathematical characteristics. A model for an element of our set of new numbers consists of a directed line segment, sometimes called a vector.

For example, directed line segments or vector models for the ordered pairs $(5, 2)$ and $(2, 5)$ are exhibited in Figure 29.1 below.

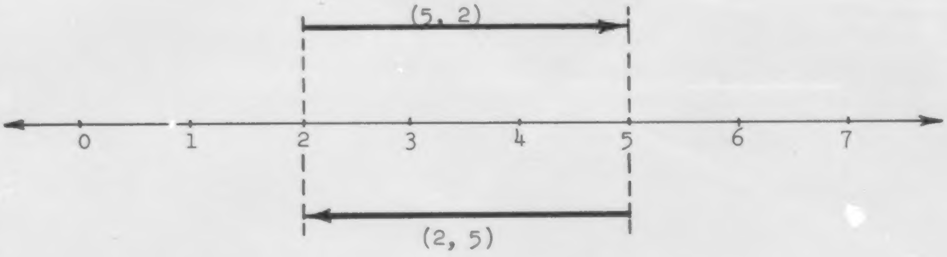
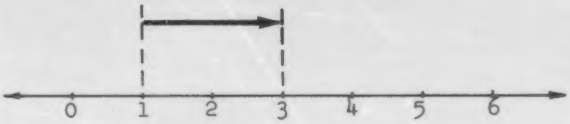


Figure 29.1

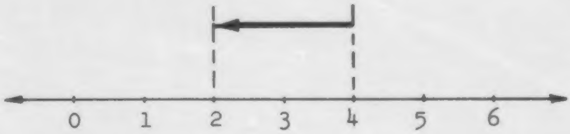
15 In the figure below, the vector model



represents the ordered pair (____, ____) or the rational number ______.

(3, 1)
$^+(3 - 1)$ or $^+2$

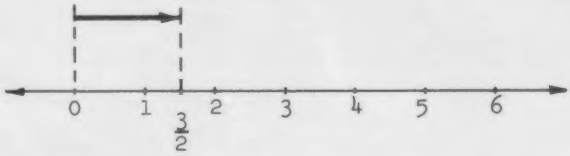
16 In the figure below, the vector model



represents the ordered pair (____, ____) or the rational number ______.

(2, 4)
$^-(4 - 2)$ or $^-2$

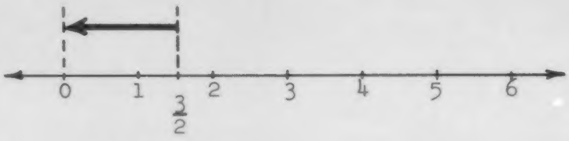
17 In the figure below, the vector model



represents the ordered pair (____, ____) or the rational number ______.

$(\frac{3}{2}, 0)$
$^+(\frac{3}{2} - 0)$ or $^+\frac{3}{2}$

18 In the figure below, the vector model



represents the ordered pair (____, ____) or the rational number ______. — (0, $\frac{3}{2}$)

$^-(\frac{3}{2} - 0)$ or $^-\frac{3}{2}$

29-2. Ordering the Set of Rational Numbers

Definition: Let (a, b) and (c, d) denote rational numbers. Then (a, b) = (c, d) if and only if a + d = b + c.

19	(5, 2) = (7, 4) since 5 + 4 = 2 + ____.	7
20	(2, 5) = (10, 13) since 2 + 13 = ____ + ____.	5 + 10
21	$(\frac{4}{2}, \frac{3}{2}) = (\frac{7}{2}, \frac{6}{2})$ since $\frac{4}{2} + \frac{6}{2} = \frac{3}{2} +$ ____.	$\frac{7}{2}$
22	$(\frac{3}{7}, \frac{2}{5}) = (\frac{3}{5}, \frac{4}{7})$ since $\frac{3}{7} + \frac{4}{7} =$ ____ + ____.	$\frac{2}{5} + \frac{3}{5}$

Definition: Let (a, b) and (c, d) denote rational numbers. Then (a, b) > (c, d) if and only if a + d > b + c.

23	(2, 3) > (4, 7) since 2 + 7 > 3 + ____.	4
24	(4, 7) > (5, 10) since 4 + 10 > ____ + ____.	7 + 5
25	$(\frac{5}{7}, \frac{2}{7}) > (\frac{8}{7}, \frac{7}{7})$ since $\frac{5}{7} + \frac{7}{7} >$ ____ + ____.	$\frac{2}{7} + \frac{8}{7}$
26	$(\frac{1}{3}, \frac{1}{4}) > (\frac{1}{5}, \frac{1}{2})$ since $\frac{1}{3} + \frac{1}{2} >$ ____ + ____.	$\frac{1}{4} + \frac{1}{5}$

Definition: Let (a, b) and (c, d) denote rational numbers. Then $(a, b) < (c, d)$ if and only if $a + d < b + c$.

27	$(10, 15) < (3, 5)$ since $10 + 5 < 15 +$ ____.	3
28	$(7, 4) < (8, 3)$ since $7 + 3 <$ ____ + ____.	4 + 8
29	$(\frac{6}{5}, \frac{8}{5}) < (\frac{4}{5}, \frac{2}{5})$ since $\frac{6}{5} + \frac{2}{5} < \frac{8}{5} +$ ____.	$\frac{4}{5}$
30	$(\frac{1}{7}, \frac{1}{5}) < (\frac{1}{2}, \frac{1}{3})$ since $\frac{1}{7} + \frac{1}{3} <$ ____ + ____.	$\frac{1}{5} + \frac{1}{2}$

Definition: Let (r, s) and (t, v) denote rational numbers. Then, one and only one of the following statements is true:

(1) $(r, s) < (t, v)$,

(2) $(r, s) = (t, v)$,

(3) $(r, s) > (t, v)$.

31 Which one of the following statements is true for (3, 5) and (7, 10)?

☐ (a) $(3, 5) < (7, 10)$

☐ (b) $(3, 5) = (7, 10)$

☐ (c) $(3, 5) > (7, 10)$

31(a) Incorrect, since $3 + 10 \nless 5 + 7$.

31(b) Incorrect, since $3 + 10 \neq 5 + 7$.

31(c) Correct, since $3 + 10 > 5 + 7$.

29-3. Addition of Rational Numbers

Definition: Let (a, b) and (c, d) denote rational numbers.
Then $(a, b) + (c, d) = (a + c,\ b + d)$.

This definition gives a computational procedure that depends only on addition of non-negative rational numbers. As with non-negative rational numbers, we call $[(a, b) + (c, d)]$ the sum of the two addends (a, b) and (c, d).

32	$(4, 5) + (7, 2) = (4 + 7,\ 5 + 2)$	
	= (____, ____)	$(11, 7)$
	or $(4, 5) + (7, 2) =$ ______.	$^+(11 - 7)$ or $^+4$
33	$(5, 2) + (2, 5) =$ (____, ____)	$(7, 7)$
	or $(5, 2) + (2, 5) =$ ______.	$(7 - 7)$ or 0
34	$(3, 6) + (5, 3) =$ (____, ____)	$(8, 9)$
	or $(3, 6) + (5, 3) =$ ______.	$^-(9 - 8)$ or $^-1$
35	$(\frac{1}{5}, \frac{1}{7}) + (\frac{3}{5}, \frac{5}{7}) =$ (____, ____)	$(\frac{4}{5}, \frac{6}{7})$
	or $(\frac{1}{5}, \frac{1}{7}) + (\frac{3}{5}, \frac{5}{7}) =$ ______.	$^-(\frac{6}{7} - \frac{4}{5})$ or $^-\frac{2}{35}$
36	$(\frac{2}{8}, \frac{4}{8}) + (\frac{5}{8}, \frac{4}{8}) =$ (____, ____)	$(\frac{7}{8}, \frac{8}{8})$
	or $(\frac{2}{8}, \frac{4}{8}) + (\frac{5}{8}, \frac{4}{8}) =$ ______.	$^-(\frac{8}{8} - \frac{7}{8})$ or $^-\frac{1}{8}$

Recall from Chapter 20 that addition in the set of non-negative rational numbers has the following properties: (1) closure; (2) commutativity; (3) associativity; (4) additive identity.

We should check to see whether or not addition as we have defined it in the set of all rational numbers has the properties characterisitc of addition in the set of non-negative rational numbers.

37	If r and s denote non-negative rational numbers, then (r + s) denotes a non-negative ______ number.	rational
38	Hence, the set of non-negative rational numbers is ______ under the operation of addition.	closed
39	If (a, b) and (c, d) are ordered pairs of non-negative rational numbers, then (a + c, b + d) ________ (is, is not) an ordered pair of non-negative rational numbers.	is
40	Since (a + c, b + d) = (a, b) + (c, d), then (a, b) + (c, d) ________ (is, is not) a rational number.	is
41	Thus, (a, b) + (c, d) is always a rational number and the set of all rational numbers is ______ under the operation of addition.	closed

We have seen that addition is closed in the set of all rational numbers as well as in the set of non-negative rational numbers.

We now turn to another property to see if it applies to addition of all rational numbers as well as to non-negative rational numbers.

42	(2, 5) + (7, 3) = (____, ____) or ______.	(9, 8) or $^{+}1$
43	(7, 3) + (2, 5) = (____, ____) or ______.	(9, 8) or $^{+}1$
44	(2, 5) + (7, 3) ________ (is, is not) equal to (7, 3) + (2, 5).	is

45	$(\frac{1}{7}, \frac{2}{9}) + (\frac{3}{7}, \frac{5}{9})$ is ____________ (equal, not equal) to $(\frac{3}{7}, \frac{5}{9}) + (\frac{1}{7}, \frac{2}{9})$.	equal
46	The order of the addends in the sum of two rational numbers ____________ (does, does not) give different results.	does not
47	The addition of two rational numbers ________ (is, is not) independent of the order in which they are added.	is
48	The examples in Frames 42 - 45 suggest the conclusion that addition in the set of all rational numbers has the ______ property.	commutative
49	A finite number of examples ________ (is, is not) sufficient to draw a general conclusion.	is not

Examples can give an intuitive justification for a generalization, but the following theorem and proof furnish conclusive evidence that the set of all rational numbers is commutative under addition.

Theorem:

$(r, s) + (t, v) = (t, v) + (r, s)$ if (r, s) and (t, v) are rational numbers.

Proof:

50	$(r, s) + (t, v) = (r + t,\ s + v)$ by the definition of ______ of rational numbers.	addition
51	$(r + t,\ s + v) = (t + r,\ v + s)$ since addition of non-negative rational numbers is ______.	commutative

52	(t + r, v + s) = (t, v) + (r, s) by the definition of addition of ______ numbers.	rational
53	Therefore, (r, s) + (t, v) = (t, v) + (r, s) is ______________ (true, not true) for any two rational numbers.	true

We have seen that the sum of two rational numbers is independent of the order of the addends. Thus, the set of all rational numbers is commutative with respect to addition.

Let us see if the result of performing two or more such additions is independent of the order in which the additions are performed.

54	[(2, 3) + (4, 1)] + (3, 7) = (6, 4) + (3, 7) = (____, ____).	(9, 11)
55	(2, 3) + [(4, 1) + (3, 7)] = (2, 3) + (7, 8) = (____, ____).	(9, 11)
56	[(2, 3) + (4, 1)] + (3, 7) ______ (=, ≠) (2, 3) + [(4, 1) + (3, 7)].	=
57	The three preceding frames seem to indicate that the sum of three rational numbers ____________ (is, is not) independent of the order of performing the addition.	is
58	[(a, b) + (c, d)] + (e, f) = (a + c, b + d) + (e, f) by the definition of addition of ______ numbers.	rational
59	(a + c, b + d) + (e, f) = ((a + c) + e, (b + d) + f) by the definition of ______ of rational numbers.	addition

60 $((a + c) + e,\ (b + d) + f) =$
$(a + (c + e),\ b + (d + f))$ by the ______ property of addition of non-negative rational numbers.

associative

61 $(a + (c + e),\ b + (d + f)) =$
$(a, b) + ((c + e), (d + f))$ by the definition of addition of ______ numbers.

rational

62 $(a, b) + ((c + e), (d + f)) =$
$(a, b) + [(c, d) + (e, f)]$ by the definition of ______ of rational numbers.

addition

63 Therefore, $[(a, b) + (c, d)] + (e, f) =$
$(a, b) + [(c, d) + (e, f)]$ ______ (is, is not) true if (a, b), (c, d) and (e, f) are rational numbers.

is

64 The statement $[(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d) + (e, f)]$ shows symbolically that addition in the set of rational numbers has

☐ (a) the closure property.

☐ (b) the commutative property.

☐ (c) the associative property.

64(a) Incorrect. While addition in the set of rational numbers has the closure property, the statement is a symbolic representation of the associative property.

64(b) Incorrect. While addition in the set of rational numbers has the commutative property, the statement is a symbolic representation of the associative property.

64(c) This response is correct. The statement

$$[(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d) + (e, f)]$$

indicates that the result of performing two successive additions is independent of their order.

We have seen that the result of performing two successive additions is independent of the order in which the additions are performed. That is, addition in the set of all rational numbers has the associative property. It is possible to verify that the result of performing any finite number of successive additions is independent of the order in which the additions are performed.

Zer[illegible]s the identity element for addition in the set of non-negative rational numbers. That is, $r + 0 = 0 + r = r$ for any non-negative rational number r.

Let us see if there is an element in the set of rational numbers which plays the role of the identity element for addition.

65	Let k denote any non-negative rational number. Then, the ordered pair (k, k) = ______.	k - k or 0
66	Since (k, k) is defined to be (k - k)= 0, it follows that (k, k) in the set of rational numbers plays the role analogous to ______ in the set of non-negative rational numbers.	zero
67	If (a, b) is any rational number, then (a, b) + (k, k) = (a + k, b + k) by the definition of ______.	addition
68	But (a + k, b + k) = (a, b) since (a + k + b) = (b + k + a) by the definition of equality of ______ numbers.	rational
69	Hence, (a, b) + (k, k) = (a, b), and (k, k) is the identity element for ______ in the set of all rational numbers.	addition

We now come to a property of addition possessed by the set of rational numbers which has not been a property of any other set of numbers considered in previous chapters.

70	Consider (2, 5) and (5, 2). (2, 5) + (5, 2) = (2 + 5, 5 + 2) = (___, ___).	(7, 7)
71	But (7, 7) = (7 - 7) = ______.	0
72	Thus, the sum of (2, 5) and (5, 2) is the rational number ______.	zero or 0
73	Consider $(\frac{2}{7}, \frac{3}{7})$ and $(\frac{3}{7}, \frac{2}{7})$. $(\frac{2}{7}, \frac{3}{7}) + (\frac{3}{7}, \frac{2}{7}) = (\frac{2}{7} + \frac{3}{7}, \frac{3}{7} + \frac{2}{7})$ = (___, ___).	$(\frac{5}{7}, \frac{5}{7})$
74	But $(\frac{5}{7}, \frac{5}{7}) = (\frac{5}{7} - \frac{5}{7})$ = ______.	0
75	Thus, the sum of $(\frac{2}{7}, \frac{3}{7})$ and $(\frac{3}{7}, \frac{2}{7})$ is the rational number ______.	zero or 0
76	Consider (t, v) and (v, t) where t and v denote non-negative rational numbers. Then, (t, v) + (v, t) = (t + v, v + t) by the definition of ______.	addition
77	But (t + v, v + t) = (t + v, t + v) by the ______ property of addition of non-negative rational numbers.	commutative
78	And (t + v, t + v) = (k, k) where k = ______.	t + v
79	Thus, (t, v) + (v, t) = (k, k), and the sum of (t, v) and (v, t) is the rational number ______.	0

The rational numbers (a, b) and (b, a) are additive inverses; (a, b) is the additive inverse of (b, a) and (b, a) is the additive inverse of (a, b). Sometimes, (a, b) is called the opposite of (b, a), and (b, a) is called the opposite of (a, b). The sum of a pair of additive inverses is always zero. This is a new property of numbers. The set of rational numbers is the first set of numbers discussed which has this property.

We illustrate models of the additive inverse elements (a, b) and (b, a) as directed line segments or vectors in Figure 29.2 below. Note that the vectors representing (a, b) and (b, a) have the same length but are in opposite directions. Assume $a > b$.

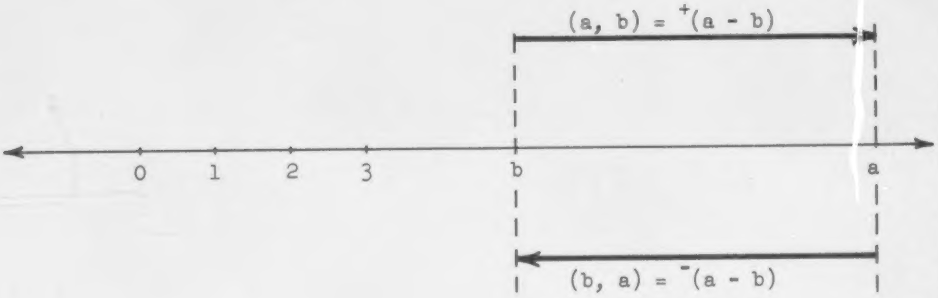


Figure 29.2

29-4. Techniques of Addition in the Set of Rational Numbers

80	$(5, 0) = {}^+(5 - \_\_\_) = {}^+5.$	0
81	$(3, 0) = {}^+(\_\_\_ - 0) = {}^+3.$	3
82	$(12, 0) = \_\_\_.$	$^+12$
83	$(0, 5) = {}^-(5 - \_\_\_) = {}^-5.$	0
84	$(0, 3) = {}^-(\_\_\_ - 0) = {}^-3.$	3
85	$(0, 12) = \_\_\_.$	$^-12$

86 Thus, $^{+}7$ = (____, 0). 7

87 And, $^{-}8$ = (0, ____). 8

In addition of rational numbers, there are three possibilities which arise. The two addends may be positive rational numbers, or negative rational numbers, or one positive and one negative.

These three cases are represented as: $^{+}r + ^{+}s$, $^{-}r + ^{-}s$, and $^{-}r + ^{+}s$. Note that $^{+}r + ^{-}s$ is basically the same problem as $^{-}r + ^{+}s$ since addition is commutative.

The following theorems furnish justification for techniques of adding rational numbers.

Theorem: If r and s are non-negative rational numbers, then $^{+}r + ^{+}s = ^{+}(r + s)$.

Proof:

88 $^{+}r + ^{+}s$ = (r, 0) + (____, ____). (s, 0)

89 (r, 0) + (s, 0) = ((r + s), (0 + 0)) by addition of ______ numbers as ordered pairs. rational

90 ((r + s), 0) = ((r + s) - ____) = (r + s). 0

91 Thus, $^{+}r + ^{+}s$ = $^{+}$(______). $^{+}(r + s)$

92 Using the above theorem, $^{+}5 + ^{+}4$ = ____. $^{+}9$

93 And, $^{+}11 + ^{+}7$ = ____. $^{+}18$

Theorem: If r and s are non-negative rational numbers, then $^{-}r + {}^{-}s = {}^{-}(r + s)$.

Proof:

94 $^{-}r + {}^{-}s = (0, r) +$ (____, ____). (0, s)

95 $(0, r) + (0, s) = (0, (r + s))$ by ______ of rational numbers as ordered pairs. addition

96 $(0, (r + s)) = {}^{-}((r + s) - 0) =$ ______. $^{-}(r + s)$

97 Thus, $^{-}r + {}^{-}s =$ ______. $^{-}(r + s)$

98 Then, $^{-}5 + {}^{-}3 = {}^{-}(5 + 3) =$ ______. $^{-}8$

99 $^{-}12 + {}^{-}7 +$ ______. $^{-}19$

Theorem: If r and s are non-negative rational numbers,

$$\text{then } {}^{-}r + {}^{+}s = \begin{cases} {}^{-}(r - s) & \text{if } r > s. \\ {}^{+}(s - r) & \text{if } r < s. \end{cases}$$

Proof:

100 $^{-}r + {}^{+}s = (0, r) +$ (____, ____). (s, 0)

101 $(0, r) + (s, 0) = (s, r)$ by addition of rational numbers as ______ ______. ordered pairs

102 $(s, r) = {}^{-}(r - s)$ if $r \underset{(<,\ =,\ >)}{\underline{\qquad}} s$. $>$

103 $(s, r) = {}^{+}(s - r)$ if $r \underset{(<,\ =,\ >)}{\underline{\qquad}} s$. $<$

104 Thus, $^{-}r + {}^{+}s = \begin{cases} \underline{\qquad} & \text{if } r > s. \\ \underline{\qquad} & \text{if } r < s. \end{cases}$ $^{-}(r - s)$ $^{+}(s - r)$

105	Then, ${}^{-}5 + {}^{+}3 = {}^{-}(5 - 3) =$ ____.	${}^{-}2$
106	${}^{-}10 + {}^{+}6 = {}^{-}(10 - 6) =$ ____.	${}^{-}4$
107	${}^{-}9 + {}^{+}3 =$ ____.	${}^{-}6$
108	${}^{-}3 + {}^{+}7 = {}^{+}(7 - 3) =$ ____.	${}^{+}4$
109	${}^{-}2 + {}^{+}5 = {}^{+}(5 - 2) =$ ____.	${}^{+}3$
110	${}^{-}4 + {}^{+}12 =$ ____.	${}^{+}8$
111	${}^{+}12 + {}^{-}4 = {}^{-}4 + {}^{+}12$ since addition of rational numbers has the ______ property.	commutative
112	${}^{+}8 + {}^{-}3 = {}^{-}3 + {}^{+}8 = {}^{+}(8 - 3) =$ ____.	${}^{+}5$
113	${}^{+}12 + {}^{-}5 = {}^{-}5 + {}^{+}12 =$ ____.	${}^{+}7$
114	${}^{+}\frac{1}{2} + {}^{+}\frac{2}{3} = {}^{+}(\frac{1}{2} + \frac{2}{3}) =$ ____.	${}^{+}\frac{7}{6}$
115	${}^{-}\frac{1}{2} + {}^{-}\frac{2}{3} = {}^{-}(\frac{1}{2} + \frac{2}{3}) =$ ____.	${}^{-}\frac{7}{6}$
116	${}^{-}\frac{1}{2} + {}^{+}\frac{2}{3} = {}^{+}(\frac{2}{3} - \frac{1}{2}) =$ ____.	${}^{+}\frac{1}{6}$
117	${}^{-}\frac{2}{3} + {}^{+}\frac{1}{2} = {}^{-}(\frac{2}{3} - \frac{1}{2}) =$ ____.	${}^{-}\frac{1}{6}$

118 ${}^{+}7 + ({}^{+}2 + {}^{-}5) =$

☐ (a) ${}^{+}14$ ☐ (b) ${}^{+}4$ ☐ (c) 0

118(a) Incorrect. ${}^{+}7 + ({}^{+}2 + {}^{-}5) = {}^{+}7 + {}^{-}3 = {}^{+}4$, not ${}^{+}14$.

118(b) Correct.

118(c) Incorrect. ${}^{+}7 + ({}^{+}2 + {}^{-}5) = {}^{+}7 + {}^{-}3 = {}^{+}4$, not 0.

29-5. Subtraction

Subtraction in the set of rational numbers is defined the same as in the previous sets of numbers.

Definition: $a - b = c$ if and only if $a = b + c$.

119	$^+5 - ^+2 = ^+3$ since $^+5 = ^+2 +$ ____.	$^+3$
120	$^-5 - ^+2 = ^-7$ since $^-5 = ^+2 +$ ____.	$^-7$
121	$^+4 - ^-1 =$ ____ since $^+4 = ^-1 + ^+5$.	$^+5$

This definition of subtraction is rather easy to handle with certain types of rational numbers but is difficult with others. The following theorem gives a procedure for subtraction which is quite effective.

Theorem: If r and s are any rational numbers, then $r - s = r + {}^-s$.

	Proof:	
122	If $r - s = n$, then $r = s + n$ from the definition of ______ of rational numbers.	subtraction
123	$r + {}^-s = r + {}^-s$ since both members of the equation are ______________ (identical, not identical).	identical
124	$r + {}^-s = (s + n) + {}^-s$ substituting $(s + n)$ for ______ in the equation of Frame 123.	r
125	$r + {}^-s = {}^-s + (s + n)$ since addition has the ______ property in rational numbers.	commutative
126	$r + {}^-s = ({}^-s + s) + n$ since addition has the ______ property in rational numbers.	associative
127	$r + {}^-s = 0 + n$ since $({}^-s + s) =$ ____.	0

128	$r + {}^-s = n$ since 0 is the ______ for addition.	identity
129	But $n = r - s$ from Frame 122. Thus, $r - s =$ ______.	$r + {}^-s$
130	$5 - 6 = 5 + {}^-6 =$ ______.	${}^-1$
131	$8 - 6 = 8 + {}^-6 =$ ______.	${}^+2$
132	${}^-6 - {}^-3 = {}^-6 + {}^-({}^-3) = {}^-6 + 3 =$ ______.	${}^-3$
133	$\frac{1}{2} - {}^-\frac{1}{3} = \frac{1}{2} +$ ______ $= \frac{5}{6}$.	$\frac{1}{3}$
134	${}^-\frac{1}{2} - {}^-\frac{1}{3} = {}^-\frac{1}{2} + \frac{1}{3} =$ ______.	${}^-\frac{1}{6}$

29-6. Multiplication of Rational Numbers

Definition: Let (a, b) and (c, d) denote rational numbers. Then,

$$(a, b) \times (c, d) = ((a \times c) + (b \times d),\ (a \times d) + (b \times c)).$$

This definition gives a computational procedure that depends only on addition and multiplication of non-negative rational numbers. As with non-negative rational numbers, we call [(a, b) × (c, d)] the product of the two factors (a, b) and (c, d).

135	$(2, 3) \times (5, 4) = ((2 \times 5) + (3 \times 4),\ (2 \times 4) + (3 \times 5))$ $= (10 + 12,\ 8 + 15)$ = (____, ____) or $(2, 3) \times (5, 4) =$ ______.	(22, 23) ${}^-(23 - 22)$ or ${}^-1$
136	$(1, 7) \times (3, 0) = ((1 \times 3) + (7 \times 0),\ (1 \times 0) + (7 \times 3))$ = (____, ____) or $(1, 7) \times (3, 0) =$ ______.	(3, 21) ${}^-(21 - 3)$ or ${}^-18$

137 $(2, 2) \times (5, 1) = ((2\times5) + (2\times1), (2\times1) + (2\times5))$

$= (\_\_\_, \_\_\_)$ (12, 12)

or $(2, 2) \times (5, 1) =$ ______. (12 - 12) or 0

Recall from Chapter 21 that multiplication in the set of non-negative rational numbers has the following properties: (1) closure; (2) commutativity; (3) associativity; (4) multiplicative identity; (5) distributivity over addition.

We now should check to see whether or not multiplication as we have defined it in the set of all rational numbers has the properties characteristic of multiplication in the set of non-negative rational numbers.

138 If a and b are non-negative rational numbers, then $(a + b)$ and $(a \times b)$ are non-negative ______ numbers. rational

139 Hence, the set of non-negative rational numbers is ______ under the operations of addition and multiplication. closed

140 If (a, b) and (c, d) are ordered pairs of non-negative rational numbers, then $((a \times c) + (b \times d), (a \times d) + (b \times c))$ ______ (is, is not) an ordered pair of non-negative rational numbers. is

141 Since $((a \times c) + (b \times d), (a \times d) + (b \times c)) = (a, b) \times (c, d)$ then $(a, b) \times (c, d)$ ______ (is, is not) a rational number. is

142 Thus, $(a, b) \times (c, d)$ is always a rational number and the set of all rational numbers is ______ under the operation of multiplication. closed

We have seen that multiplication is closed in the set of all rational numbers as well as in the set of non-negative rational numbers.

We turn now to another property to see if it applies to multiplication of all rational numbers as well as to non-negative rational numbers.

143	(1, 4) × (5, 2) = (____, ____).	(13, 22)
144	(5, 2) × (1, 4) = (____, ____).	(13, 22)
145	(1, 4) × (5, 2) ________ (is, is not) equal to (5, 2) × (1, 4).	is
146	(4, 0) × (2, 5) is __________ (equal, not equal) to (2, 5) × (4, 0).	equal
147	The order of the factors in the product of two rational numbers __________ (does, does not) give different results.	does not
148	The two examples given in Frames 143 - 146 suggest the conclusion that multiplication in the set of all rational numbers has the ______ property.	commutative
149	A finite number of examples ________ (is, is not) sufficient to draw a general conclusion.	is not

A finite number of examples can give an intuitive justification for a generalization, but the following theorem and proof furnish conclusive evidence that multiplication in the set of all rational numbers is commutative.

Theorem:

$(r, s) \times (t, v) = (t, v) \times (r, s)$, if
(r, s) and (t, v) are rational numbers.

Proof:

150 $(r, s) \times (t, v) = ((r \times t) + (s \times v), (r \times v) + (s \times t))$
by the definition of ______ of rational numbers. multiplication

151 $((r \times t) + (s \times v), (r \times v) + (s \times t)) =$
$((t \times r) + (v \times s), (v \times r) + (t \times s))$
since multiplication of non-negative rational
numbers is ______. commutative

152 $((t \times r) + (v \times s), (v \times r) + (t \times s)) =$
$((t \times r) + (v \times s), (t \times s) + (v \times r))$
since addition of non-negative rational numbers
is ______. commutative

153 $((t \times r) + (v \times s), (t \times s) + (v \times r)) =$
$(t, v) \times (r, s)$ by the definition of
multiplication of ______ numbers. rational

154 Therefore, $(r, s) \times (t, v) = (t, v) \times (r, s)$
is ______ for any two rational numbers. true
(true, not true)

We have seen that the product of two numbers is independent of the order of the factors for all rational numbers as well as for non-negative rational numbers.

Let us see if the result of performing two or more successive multiplications is independent of the order in which the multiplications are performed.

155 $[(2, 1) \times (5, 3)] \times (3, 0)$
$= (\_\_\_, \_\_\_) \times (3, 0)$ (13, 11)
$= (\_\_\_, \_\_\_)$. (39, 33)

156 (2, 1) × [(5, 3) × (3, 0)]

= (2, 1) × (____, ____) (15, 9)

= (____, ____). (39, 33)

157 [(2, 1) × (5, 3)] × (3, 0) ________ =
(=, ≠)
(2, 1) × [(5, 3) × (3, 0)].

158 [(1, 4) × (2, 5)] × (0, 3) ________ =
(=, ≠)
(1, 4) × [(2, 5) × (0, 3)].

159 The preceding frames seem to indicate that multiplication in the set of all rational numbers ______________ have the associative property. does
(does, does not)

In the following frames, we exhibit a proof of the associative property of multiplication in the set of all rational numbers. Due to the length of the proof and the mathematical rigor involved, the reader may wish to proceed directly to Frame 168 and in so doing will not lose continuity in the development of properties of the set of all rational numbers.

In the following theorem and proof, we continue to use the symbol × to denote multiplication in the set of all rational numbers. However, we will not use this symbol to denote multiplication in the set of non-negative rational numbers. We will write the symbol ab to denote the product a × b where a and b denote members of the set of non-negative rational numbers.

Theorem:

[(a, b) × (c, d)] × (e, f) = (a, b) × [(c, d) × (e, f)]
if (a, b), (c, d) and (e, f) are rational numbers.

Proof:

160 [(a, b) × (c, d)] × (e, f) = (ac + bd, ad + bc) × (e, f)
by the definition of multiplication of ______ numbers. rational

161 (ac + bd, ad + bc) × (e, f) =
((ac + bd)e + (ad + bc)f, (ac + bd)f + (ad + bc)e)
by the definition of ______ of rational numbers.

multiplication

162 ((ac + bd)e + (ad + bc)f, (ac + bd)f + (ad + bc)e) =
((ace + bde) + (adf + bcf), (acf + bdf) + (ade + bce))
by the distributive property of non-negative ______ numbers.

rational

163 ((ace + bde) + (adf + bcf), (acf + bdf) + (ade + bce)) =
((ace + adf) + (bcf + bde), (acf + ade) + (bce + bdf))
by the commutative and associative properties of addition of non-negative ______ numbers.

rational

164 ((ace + adf) + (bcf + bde), (acf + ade) + (bce + bdf)) =
(a(ce + df) + b(cf + de), a(cf + de) + b(ce + df))
by the ______ property of multiplication with respect to addition in the set of non-negative rational numbers.

distributive

165 (a(ce + df) + b(cf + de), a(cf + de) + b(ce + df)) =
(a, b) × (ce + df, cf + de)
by the definition of multiplication of ______ numbers.

rational

166 (a, b) × (ce + df, cf + de) =
(a, b) × [(c, d) × (e, f)] by the definition of ______ of rational numbers.

multiplication

167 Therefore,

$$[(a, b) \times (c, d)] \times (e, f) \underset{(=,\ \neq)}{\underline{\qquad}} (a, b) \times [(c, d) \times (e, f)].$$

=

168 The statement

$$[(a, b) \times (c, d)] \times (e, f) = (a, b) \times [(c, d) \times (e, f)]$$

shows symbolically that multiplication in the set of all rational numbers has

☐ (a) the closure property

☐ (b) the commutative property

☐ (c) the associative property

168(a) Incorrect. While multiplication in the set of all rational numbers has the closure property, the statement is a symbolic representation of the associative property.

168(b) Incorrect. While multiplication in the set of all rational numbers has the commutative property, the statement is a symbolic representation of the associative property.

168(c) Correct. The statement

$$[(a, b) \times (c, d)] \times (e, f) = (a, b) \times [(c, d) \times (e, f)]$$

indicates that the result of performing two successive multiplications is independent of their order.

We have seen that the result of performing two successive multiplications is independent of the order in which the multiplications are performed. That is, multiplication in the set of all rational numbers has the associative property. It is possible to verify that the result of performing any finite number of successive multiplications is independent of the order in which the multiplications are performed.

One is the identity element for multiplication in the set of non-negative rational numbers. That is, $1 \times r = r \times 1 = r$ for any non-negative rational number r.

169	The set of non-negative rational numbers ______ (does, does not) have an element which is the identity for multiplication.	does
170	(2, 5) × (1, 0) = (___, ___).	(2, 5)
171	(2, 5) × (2, 1) = (9, 12) = (2, 5) since $^{-}(12 - 9)$ = ______.	$^{-}(5 - 2)$
172	(2, 5) × (3, 2) = (16, 19) = (____, ____) since $^{-}(19 - 16) = {}^{-}(5 - 2)$.	(2, 5)

173 The identify for multiplication in the set of all rational numbers is represented by the ordered pair

☐(a) (k, k) ☐(b) (r + 1, r) ☐(c) (1, 0)

173(a) Incorrect. (k, k) is the identity for addition, not multiplication.

173(b) Correct. Proceed to Frame 178.

173(c) Correct. (1, 0) is another name for (r + 1, r). See 173(b) and proceed to Frame 178.

174	(a, b) × (r + 1, r) = (a × (r + 1) + (b × r), (a × r) + (b × (r + 1)) by definition of ______ of rational numbers.	multiplication
175	(a × (r + 1) + (b × r), (a × r) + b × (r + 1)) = ((a × r) + (a × 1) + (b × r), (a × r) + (b × r) + (b × 1)) by the ______ property of multiplication with respect to addition in the set of non-negative rational numbers.	distributive

176	$((a\times r)+(a\times 1)+(b\times r),\ (a\times r)+(b\times r)+(b\times 1)) =$ $((a\times r)+a+(b\times r),\ (a\times r)+(b\times r)+b)$ by the multiplication property of ______ in the set of non-negative rational numbers.	1 or one
177	$((a\times r)+a+(b\times r),\ (a\times r)+(b\times r)+b) = (a, b)$ since $(a\times r)+a+(b\times r)+b = (a\times r)+(b\times r)+b+a$ by the definition of equality of ______ numbers.	rational
178	Hence, $(a, b) \times (r+1, r) = (a, b)$ and the identity for ______ in the set of all rational numbers is $(r+1, r)$.	multiplication

Since $r + 1 > r$ for any non-negative rational number r, the identity for multiplication, denoted by $(r + 1, r)$, is equal to

$$^+(r + 1 - r) = {}^+(r - r + 1)$$
$$= {}^+(0 + 1)$$
$$= {}^+1$$ which is read positive one.

Hence, the identity for multiplication in the set of all rational numbers may be denoted by the symbol $^+1$.

We exhibit a directed line segment model of positive one in Figure 29.3 below.

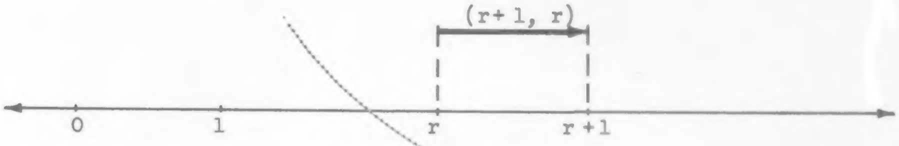


Figure 29.3

Recall that multiplication is distributive over addition in the set of non-negative rational numbers. That is, $r \times (t + v) = (r \times t) + (r \times v)$. Let us see whether or not multiplication is distributive over addition in the set of all rational numbers.

Theorem: $(a, b) \times [(c, d) + (e, f)] = [(a, b) \times (c, d)] + [(a, b) \times (e, f)]$
if (a, b), (c, d) and (e, f) are rational numbers.

In the following frames, we exhibit a proof of the above theorem. Due to the length of the proof and the mathematical rigor involved, the reader may choose to proceed directly to Frame 187 and in so doing will not lose continuity in the development of properties of the set of all rational numbers.

In the following proof of the distributive property of multiplication over addition, we continue to use the symbol $\times$ to denote multiplication in the set of all rational numbers. However, we will not use this symbol to denote multiplication in the set of non-negative rational numbers. We will write the symbol ab to denote the product $a \times b$ where a and b denote members of the set of non-negative rational numbers.

179 $(a, b) \times [(c, d) + (e, f)] = (a, b) \times ((c + e), (d + f))$
by the definition of ______ of rational numbers.

addition

180 $(a, b) \times ((c + e), (d + f)) =$
$(a(c + e) + b(d + f),\ a(d + f) + b(c + e))$
by the definition of ______ of rational numbers.

multiplication

181 $(a(c + e) + b(d + f),\ a(d + f) + b(c + e)) =$
$((ac + ae) + (bd + bf),\ (ad + af) + (bc + be))$
since multiplication is ______ over addition
in the set of non-negative rational numbers.

distributive

182 $((ac + ae) + (bd + bf),\ (ad + af) + (bc + be)) =$
$((ac + bd) + (ae + bf),\ (ad + bc) + (af + be))$
by the commutative and associative properties of
addition of non-negative ______ numbers.

rational

183 $((ac + bd) + (ae + bf),\ (ad + bc) + (af + be)) =$
$((ac + bd),\ (ad + bc)) + ((ae + bf),\ (af + be))$
by the definition of ______ of rational numbers.

addition

184 $((ac + bd), (ad + bc)) + ((ae + bf), (af + be)) =$
$[(a, b) \times (c, d)] + [(a, b) \times (e, f)]$ by the definition of ______ of rational numbers.

multiplication

185 Therefore,
$(a, b) \times [(c, d) + (e, f)] =$
$[(a, b) \times (c, d)] + [(a, b) \times (e, f)]$
is ______ for any three rational numbers.
(true, not true)

true

186 Hence, in the set of all rational numbers, multiplication is ______ over addition.

distributive

187 $(1, 2) \times [(3, 0) + (4, 0)] = (1, 2) \times (7, 0)$
$= (\_\_\_, \_\_\_).$

(7, 14)

188 $(1, 2) \times [(3, 0) + (4, 0)]$
$= [(1, 2) \times (3, 0)] + [(1, 2) \times (4, 0)]$
$= (3, 6) + (4, 8)$
$= (\_\_\_, \_\_\_).$

(7, 14)

189 $(1, 0) \times [(2, 3) + (5, 5)]$
$= [(1, 0) \times (2, 3)] + [(1, 0) \times (5, 5)]$
$= (2, 3) + (\_\_\_, \_\_\_)$
$= (\_\_\_, \_\_\_)$ or ______ .

(5, 5)
(7, 8) or $^{-}1$

190 $(3, 0) \times [(\frac{5}{2}, \frac{1}{2}) + (\frac{7}{2}, \frac{9}{2})]$
$= [(3, 0) \times (\frac{5}{2}, \frac{1}{2})] + [(3, 0) \times (\frac{7}{2}, \frac{9}{2})]$
$= (\frac{15}{2}, \frac{3}{2}) + (\frac{14}{2}, \frac{27}{2}) = (\_\_\_, \_\_\_)$ or ______.

$(\frac{29}{2}, \frac{30}{2})$ or $^{-}\frac{1}{2}$

In the set of non-negative rational numbers, the identity element for addition, namely zero, has the following multiplication property:

$$0 \times r = r \times 0 = 0 .$$

191 $(2, 7) \times (1, 1) = (2 + 7, \; 2 + 7)$

$= (\_\_\_, \_\_\_)$ or $\_\_\_\_\_$. (9, 9) or 0

192 $(2, 7) \times (5, 5) = (10 + 35, \; 10 + 35)$

$= (\_\_\_, \_\_\_)$ or $\_\_\_\_\_$. (45, 45) or 0

193 $(2, 7) \times (k, k) = ((2 \times k) + (7 \times k), (2 \times k) + (7 \times k))$

$= (9 \times k) - \_\_\_\_\_)$ $9 \times k$

$= \_\_\_\_\_.$ 0

194 $(t, v) \times (k, k) = ((t \times k) + (v \times k), \; (t \times k) + (v \times k))$
by the definition of ______ of rational numbers. multiplication

195 $((t \times k + (v \times k), \; (t \times k) + (v \times k)) =$
$[(t \times k) + (v \times k)] - [(t \times k) + (v \times k)] = 0$
by the definition of ______ numbers. rational

196 Hence, $(k, k) \times (t, v)$ is zero and the product of the identity for addition and the rational number (t, v) yields the identity for ______. addition

Recall that in the set of non-negative rational numbers each member other than 0 has a multiplicative inverse (or reciprocal). For example, $\frac{5}{2}$ and $\frac{2}{5}$ are reciprocals and their product $\frac{5}{2} \times \frac{2}{5}$ is one.

We exhibit a proof that each element (except the zero element) in the set of all rational numbers has a multiplicative inverse.

Case I: Let t denote a non-zero non-negative rational number and consider $(t, 0)$. Then $(t, 0) = {}^{+}(t - 0) = {}^{+}t$ is a positive rational number.

197 $(t, 0) \times (\frac{1}{t}, 0) = ((t \times \frac{1}{t}), 0)$ by the definition of ______ of rational numbers. multiplication

198 Since $t \times \frac{1}{t} = 1$, then $((t \times \frac{1}{t}), 0) = (1, 0)$ or ______.

$^+1$

199 Hence, the product $(t, 0) \times (\frac{1}{t}, 0)$ is $^+1$ and the rational numbers $(t, 0)$ and $(\frac{1}{t}, 0)$ are ______.

reciprocals

200 $(7, 0)$ and $(\frac{1}{7}, 0)$ are reciprocals since $(7, 0) \times (\frac{1}{7}, 0) = (1, 0)$ or ______.

$^+1$

201 $(\frac{3}{1}, 0$ and $(\frac{1}{3}, 0$ are reciprocals since $(\frac{3}{1}, 0) \times (\frac{1}{3}, 0) = (1, 0)$ or ______.

$^+1$

Case II: Let t denote a non-zero non-negative rational number and consider $(0, t)$. Then $(0, t) = {}^-(t - 0) = {}^-t$ is a negative rational number.

202 $(0, t) \times (0, \frac{1}{t}) = ((t \times \frac{1}{t}), 0)$ by the definition of ______ of rational numbers.

multiplication

203 Since $t \times \frac{1}{t} = 1$, then $((t \times \frac{1}{t}), 0) = (\_\_\_\_, \_\_\_\_)$ or $^+1$.

$(1, 0)$

204 Hence, the product $(0, t) \times (0, \frac{1}{t})$ is ______ and the rational numbers $(0, t)$ and $(0, \frac{1}{t})$ are reciprocals.

$^+1$

205 $(0, 5)$ and $(0, \frac{1}{5})$ are reciprocals since $(0, 5) \times (0, \frac{1}{5}) = (1, 0)$ or ______.

$^+1$

206 $(0, \frac{2}{5})$ and $(0, \frac{5}{2})$ are reciprocals since $(0, \frac{2}{5}) \times (0, \frac{5}{2}) = (1, 0)$ or ______.

$^+1$

207 The reciprocal of (0, 1) is

☐(a) (1, 0) ☐(c) (0, 0)

☐(b) (0, 1) ☐(d) (1, 1)

207(a) Incorrect. $(1, 0) \times (0, 1) = (0, 1) = {}^-1$ not ${}^+1$.

207(b) Correct. $(0, 1) \times (0, 1) = (1, 0) = {}^+1$.

207(c) Incorrect. $(0, 0) \times (0, 1) = (0, 0) = 0$ not ${}^+1$.

207(d) Incorrect. $(1, 1) \times (0, 1) = (1, 1) = 0$ not ${}^+1$.

29-7. Techniques of Multiplication in the Set of All Rational Numbers

In multiplication of rational numbers there are three possibilities which arise. The two factors may be positive rational numbers, or negative rational numbers, or one positive and one negative. The three cases are covered in the three theorems which follow.

Theorem: If r and s are non-negative rational numbers, then ${}^+r \times {}^+s = {}^+(r \times s)$.

	Proof:	
208	${}^+r \times {}^+s = (r, 0) \times$ (____, ____).	(s, 0)
209	$(r, 0) \times (s, 0) = ((r \times s) + (0 \times 0), (r \times 0) + (s \times 0))$ by multiplication of ______ numbers as ordered pairs.	rational
210	$((r \times s), 0) = {}^+((r \times s) - 0) =$ ______.	${}^+(r \times s)$
211	Thus, ${}^+r \times {}^+s =$ ______.	${}^+(r \times s)$
212	${}^+7 \times {}^+3 = {}^+(7 \times 3) = {}^+$______.	${}^+21$

213	$^+\frac{1}{2} \times {}^+\frac{2}{5} = {}^+(\frac{1}{2} \times \frac{2}{5}) =$ ______.	$^+\frac{2}{10}$ or $^+\frac{1}{5}$

Theorem: If r and s are non-negative rational numbers, then $^-r \times {}^-s = {}^+(r \times s)$.

	Proof:	
214	$^-r \times {}^-s = (0, r) \times (\_\_\_, \_\_\_)$.	$(0, s)$
215	$(0, r) \times (0, s) = ((0 \times 0) + (r \times s), (0 \times s) + (r \times 0))$ by ______ of rational numbers as ordered pairs.	multiplication
216	$((r \times s), 0) = {}^+((r \times s) - 0) = {}^+(\_\_\_\_\_\_)$.	$r \times s$
217	Thus, $^-r \times {}^-s = {}^+(\_\_\_\_\_\_)$.	$r \times s$
218	$^-5 \times {}^-2 = {}^+(5 \times 2) =$ ______.	$^+10$
219	$^-\frac{2}{3} \times {}^-\frac{4}{3} = {}^+(\frac{2}{3} \times \frac{4}{3}) =$ ______.	$^+\frac{8}{9}$

Theorem: If r and s are non-negative rational numbers, then $^-r \times {}^+s = {}^-(r \times s)$.

	Proof:	
220	$^-r \times {}^+s = (0, r) \times (\_\_\_, \_\_\_)$.	$(s, 0)$
221	$(0, r) \times (s, 0) = ((0 \times s) + (r \times 0), (0 \times 0) + (r \times s))$ by multiplication of rational numbers as ______ pairs.	ordered
222	$(0, (r \times s)) = {}^-((r \times s) - 0) =$ ______.	$^-(r \times s)$

223	Thus, $^-r \times {}^+s =$ ______.	$^-(r \times s)$
224	$^-3 \times {}^+4 = {}^-(3 \times 4) =$ ______.	$^-12$
225	$^-\frac{1}{2} \times {}^+\frac{3}{5} = {}^-(\frac{1}{2} \times \frac{3}{5}) =$ ______.	$^-\frac{3}{10}$
226	$\frac{4}{3} \times {}^-\frac{2}{5} = {}^-\frac{2}{5} \times \frac{4}{3} =$ ______.	$^-\frac{8}{15}$

227 $^-\frac{1}{2} \times (^-\frac{1}{3} \times {}^-\frac{1}{5}) =$

☐(a) $\frac{1}{30}$ ☐(b) $^-\frac{1}{30}$ ☐(c) $^-\frac{3}{30}$

227(a) Incorrect. $^-\frac{1}{2} \times (^-\frac{1}{3} \times {}^-\frac{1}{5}) = {}^-\frac{1}{2} \times {}^+\frac{1}{15} = {}^-\frac{1}{30}$.

227(b) Correct.

227(c) Incorrect. $^-\frac{1}{2} \times (^-\frac{1}{3} \times {}^-\frac{1}{5}) = {}^-\frac{1}{2} \times {}^+\frac{1}{15} = {}^-\frac{1}{30}$.

29-8. Division

Division by a non-zero rational number is defined as follows:

Definition: $r \div s = n$ if and only if $r = s \times n$ provided $s \neq 0$.

Division is made easier by the following theorem:

Theorem: $r \div s = r \times \frac{1}{s}$ where $\frac{1}{s}$ is the reciprocal or multiplicative inverse of s and $s \times \frac{1}{s} = 1$.

Proof:

228	Let $r \div s = n$. Then $r = s \times n$ by the definition of ______ of rational numbers.	division
229	$r \times \frac{1}{s} = r \times \frac{1}{s}$ since both members of the equation ______ (are, are not) identical.	are

230	$r \times \frac{1}{s} = (s \times n) \times \frac{1}{s}$ since $(s \times n)$ is the same as ______ from Frame 228.	r
231	$r \times \frac{1}{s} = \frac{1}{s} \times (s \times n)$ since multiplication has the ______ property in rational numbers.	commutative
232	$r \times \frac{1}{s} = (\frac{1}{s} \times s) \times n$ since multiplication has the ______ property in rational numbers.	associative
233	$r \times \frac{1}{s} = 1 \times n$ since $\frac{1}{s} \times s =$ ______.	1
234	$r \times \frac{1}{s} =$ ______ since 1 is the multiplicative identity.	n
235	From Frame 228, $n =$ ______.	$r \div s$
236	Thus, $r \div s = r \times \frac{1}{s}$ where $\frac{1}{s}$ is the ______ of s.	reciprocal or multiplicative inverse
237	$5 \div {}^-3 = \frac{5}{1} \times \frac{{}^-1}{3} =$ ______.	$\frac{{}^-5}{3}$
238	${}^-8 \div {}^-4 = \frac{{}^-8}{1} \times \frac{{}^-1}{4} =$ ______.	$\frac{{}^+8}{4}$ or ${}^+2$
239	$\frac{{}^-2}{3} \div \frac{1}{2} = \frac{{}^-2}{3} \times \frac{2}{1} =$ ______.	$\frac{{}^-4}{3}$
240	$\frac{7}{9} \div \frac{{}^-1}{2} = \frac{7}{9} \times \frac{{}^-2}{1} =$ ______.	$\frac{{}^-14}{9}$

CHAPTER 30

THE REAL NUMBERS

We are now ready to make the last extension (for us) of the number system. In the first twelve chapters our concern was the whole numbers, their operations and properties. In Chapters 18 - 24 the system of non-negative rational numbers was developed and the operations on these numbers and the properties of the operations were studied. At the time we called this set of numbers the rational numbers although more accurately, we should have called them, as we did just now, the non-negative rationals. In Chapter 29 we developed the complete system of rational numbers including the negative numbers and studied their operations and properties. Remember that now "rational number" refers to any such number as:

$$^{+}2,\quad ^{-}3,\quad ^{-}\frac{2}{3},\quad 0,\quad ^{+}1.6,\quad ^{-}2.34,\quad ^{-}\frac{15}{3},\quad ^{+}17, \text{ et cetera.}$$

From now on we shall almost always write the positive rationals omitting the $^{+}$ superscript. When a letter such as a or r is used to represent a rational number, it should be understood that it may represent 0 or a negative rational, just as well as a positive one.

30-1. Properties of Operations on the Rational Numbers

1	If a and b are any rational numbers, the fact that a + b also is a rational number illustrates the ______ property of rational numbers under addition.	closure
2	If a and b are any rational numbers, the fact that a × b also is a rational number illustrates the closure property of rational numbers under ______.	multiplication
3	If a and b are any rational numbers, the statement a + b = b + a is a symbolic representation of the ______ property of rational numbers for addition.	commutative

4	Multiplication of rational numbers is a commutative operation. The symbolic representation of this statement is $a \times b =$ ______.	$b \times a$
5	If a, b and c are any rational numbers, then $(a + b) + c = a + (b + c)$ is a symbolic representation of the associative property of rational numbers with respect to ______.	addition
6	If a, b and c are any rational numbers, then $(a \times b) \times c = a \times (b \times c)$ is a symbolic representation of the associative property of multiplication of ______ numbers.	rational
7	The distributive property relates the operations of multiplication and addition. Thus, if a, b and c are rational numbers, a symbolic representation of the distributive property of rational numbers, that is, multiplication distributes over addition, is the statement $a \times (b + c) =$ ____ + ____.	$(a \times b) + (a \times c)$
8	There is a rational number denoted by 0 such that $a + 0 = 0 + a = a$ for any rational number a. Thus, 0 is called the identity for ______ of rational numbers.	addition
9	There is a rational number denoted by 1 such that $a \times 1 = 1 \times a = a$ for any rational number a. Thus, 1 is called the ______ for multiplication of rational numbers.	identity
10	If a is any rational number, then there is another rational number b such that $a + b = 0$. Usually b is written as ^{-}a and $a + {^{-}a} =$ _____. The rational numbers a and ^{-}a are called additive inverses or opposites.	0

11	The existence of additive inverses makes it possible to convert any subtraction problem into an ______ problem. Addition and subtraction are called inverse operations.	addition
12	Thus, $a - b = a +$ _____.	^{-}b
13	Hence, $a - b$ is always a rational number and the rational numbers are closed under the operation of ______	subtraction
14	If a is any non-zero rational number, then there is a rational number b such that $a \times b = 1$. The number b is sometimes written as $\frac{1}{a}$ so that $a \times \frac{1}{a} =$ _____. The rational numbers a and $\frac{1}{a}$ are called multiplicative inverses or reciprocals.	1
15	The existence of multiplicative inverses makes it possible to convert any division problem into a ______ problem. Division and multiplication are called inverse operations.	multiplication
16	Thus, $a \div b = a \times$ _____.	$\frac{1}{b}$
17	Hence $a \div b$, where $b \neq 0$, is always a rational number and the rational numbers are closed under the operation of ______ with the exception of division by zero.	division

18 If a and b are any rational numbers and a - b is a positive rational number, then

☐(a) a = b ☐(b) a < b ☐(c) a > b

18(a) Incorrect. If a = b, then a - b = 0 and 0 is not a positive rational number. See 18(c).

18(b) Incorrect. If a < b, then a - b is a negative rational number, not a positive one. See 18(c).

18(c) Correct. If a > b, then a - b is a positive rational number. a > b means that a positive rational number p may be found such that a = b + p.

19 Between any two distinct rational numbers there is at least a third rational number. This property is called the d______y property.

density

30-2. A Number Line for the Set of Rational Numbers

When we studied whole numbers and the non-negative rational numbers, we found that a good physical model such as a number line was a great help to our understanding.

To exhibit a number line for the set of all rational numbers, we proceed as follows:

Select a point on a line to represent zero which we call the origin and label it with 0. We then choose a direction on the line for the positive direction and a unit length. The point one unit from 0 in the positive direction is labeled with $^{+}1$, the point two units in the positive direction from 0 is labeled with $^{+}2$, and so on. In general, any point which in Chapter 18 was labeled with the number $\frac{a}{b}$ may now be labeled with $^{+}\frac{a}{b}$. See Figure 30.1 below.

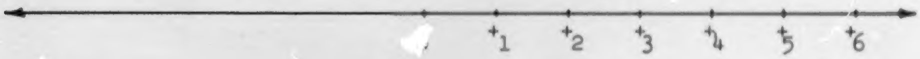


Figure 30.1

So far the points of our number line are on one of the rays from 0 and represent only the positive rational numbers and zero. But now, using the ray in the opposite direction from 0, and calling it the negative direction, we can represent the negative rational numbers by points on it.

Thus, we label the point on unit in the negative direction with $^{-}1$, the point two units in the negative direction with $^{-}2$, the point half-way between with $^{-}\frac{3}{2}$, and so on. See Figure 30.2 below.

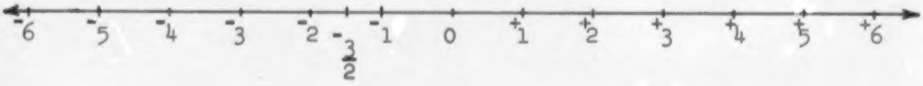


Figure 30.2

When using the number line, we sometimes talk loosely about the "point" or the "number" rather than "the point representing the number." This may be confusing, but it certainly saves time and effort.

Figure 30.2 makes it apparent that the positive rational numbers may be thought of as extending indefinitely to the right of 0, and the negative rational numbers indefinitely to the left of 0.

In Chapter 19 it was shown that if any two rational numbers are given, there is always another rational number between them. Another way of saying this is that if r is a rational number, there is no next larger one. This property of rational numbers is called density.

From this it follows that there are many rational numbers and corresponding to them on the number line, many "rational points." Moreover, the points are spread throughout the number line. Any segment, no matter how small, contains infinitely many rational points. One might think that all the points on the number line are rational points, that is, that every point on the line corresponds to a rational number. This is not so. There are many points on the line that are not associated with rational numbers.

30-3. Irrational Numbers

In Figure 30.3 below, ABCD is a square.

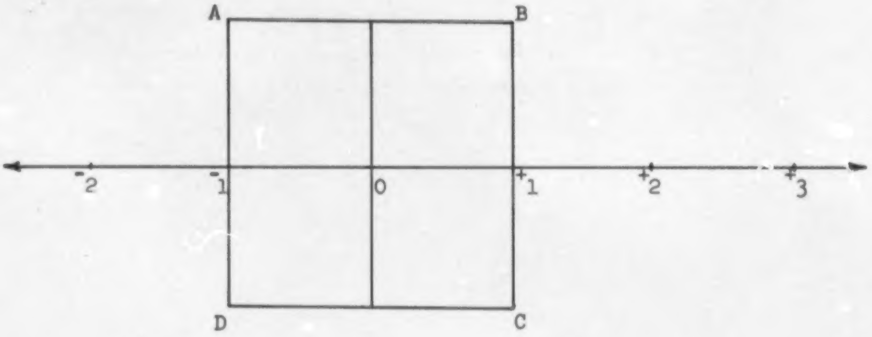


Figure 30.3

20	Each side of the square region ABCD has length ______.	2
21	The area of the square region ABCD is $2 \times 2 =$ ______.	4
22	The square region ABCD is partitioned into 4 congruent sub-regions, each with area equal to ______.	1

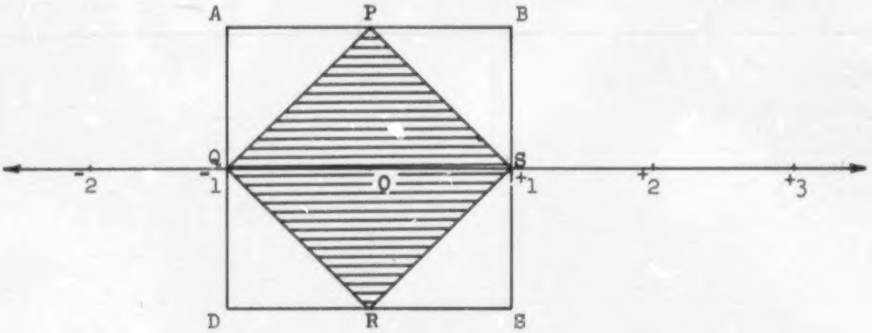
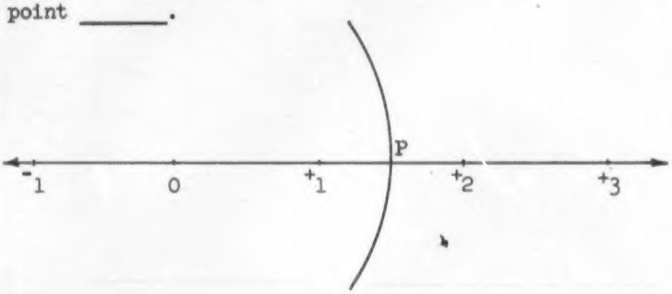


Figure 30.4

23	In Figure 30.4 each of the four congruent subregions of Figure 30.3 has been partitioned into two congruent triangular regions each of area ______.	$\frac{1}{2}$
24	Hence the area of region PQRS is $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} =$ ______.	$\frac{4}{2}$ or 2
25	PQRS is a square and $\overline{PQ} \cong \overline{QR} \cong \overline{RS} \cong$ ______.	$\overline{SP}$ or $\overline{PS}$
26	$m(\overline{PQ}) = m(\overline{QR}) = m(\overline{RS}) =$ ______.	$m(\overline{SP})$ or $m(\overline{PS})$
27	Let $m(\overline{PS}) = s$. Then, $s \times s = 2$ by the definition of ______ of a square region.	area
28	On the rational number line below, draw an arc of a circle with center at point 0 and radius of $m(\overline{PS}) = s$ intersecting the number line at point ______.  	P
29	The point P corresponds to the number ______.	s

30 The number s corresponding to the point P is

☐ (a) a rational number

☐ (b) not a rational number

☐ (c) I don't know

30(a) Incorrect. See 30(b) and proceed to the next frame.

30(b) Correct. Proceed to the next frame.

30(c) Possibly a correct response. See 30(b) and proceed to the next frame.

To prove that s is not a rational number indirect reasoning will be used. It will be assumed that s is a rational number and then it will be shown that this assumption leads to an impossible conclusion.

Theorem: If $s \times s = 2$, then s is not a rational number.

Proof:

31	Assume $s = \frac{p}{q}$ with p a member of $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}$ and q a member of $\{1, 2, 3, 4, \ldots\}$ and $\frac{p}{q}$ is in lowest form. Thus, we have assumed that s is a ______ number.	rational
32	Since $\frac{p}{q}$ is in lowest form, the only common factor of p and q is ______.	1
33	Since $s = \frac{p}{q}$, $s \times s = \frac{p}{q} \times \frac{p}{q} =$ ______.	2
34	$\frac{p}{q} \times \frac{p}{q} = \frac{p \times p}{q \times q}$ by the definition of ______ of rational numbers.	multiplication
35	Since $\frac{p \times p}{q \times q} = 2 = \frac{2}{1}$, then $(p \times p) \times 1 = (q \times q) \times$ _____ by the order property of rational numbers.	2

36	Thus, $p \times p = (q \times q) \times 2$ and $p \times p$ is an ______ (odd, even) number.	even
37	Since $p \times p$ is an even number, then p is an ______ (odd, even) number.	even
38	Every even number can be written in the form $2 \times k$ for some whole number k. Thus $p =$ ______.	$2 \times k$
39	Now $p \times p = (2 \times k) \times (2 \times k) =$ ____ $\times (k \times k)$.	4
40	Since $p \times p = (q \times q) \times 2$, then $4 \times (k \times k) = 2 \times (q \times q)$ and dividing both members of this equation by 2 gives $2 \times (k \times k) =$ ______.	$q \times q$
41	Since $2 \times (k \times k) = q \times q$, the number $q \times q$ is an ______ (odd, even) number.	even
42	If $q \times q$ is even, then q is ______ (odd, even).	even
43	Thus from Frame 37 p is even and from Frame 42 q is even and p and q have the common factor ______.	2
44	Since $\frac{p}{q}$ is in lowest form according to Frame 31, p and q have only 1 as a common factor. Thus Frame 31 ______ (is, is not) a contradiction of Frame 43.	is
45	Therefore, the assumption made in Frame 31 is not true and as a consequence, the number s ______ (is, is not) a rational number.	is not

Thus $\underline{s}$, where $s \times s = 2$, is *not* a rational number, but the point P in Frame 28 is a perfectly definite point on the number line.

We now have a point on the number line and *no* rational number to associate with it. We simply assert that there is an irrational number $\underline{s}$ associated with this point. Since $s \times s = 2$ and the radical sign $\sqrt{\ }$ is used for square roots, we give the name $\sqrt{2}$ to this number.